Representation of electrical signals by a series of exponential terms

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Abstract— **This study is devoted to the representation of electrical signals by a finite exponential representation. Three methods are investigated. The first one is Prony method which consists in a time domain estimation. The second one is equivalent in the frequency domain. The third one is a new method which consists in a frequency domain estimation permitting to limit the size of the model. These methods are analyzed on examples.**

Keywords: Prony method, characteristic impedance.

Notation

t: time
y: analytical signal
y_n: discrete form of the signal y
x_n: sample of the discrete signal y_n
α_k: poles of the kth component of Prony expansion
A_k: residue associated to the pole α_k
N: order of Prony decomposition
P_N: linear prediction polynomial
H: $IFFT$ (Inverse Fast Fourier Transform) of x_n
M: size of the signal x_n
Δt, Δ: sampling step
LSME: least square mean estimation

I. Introduction

It is of considerable interest to use a complex exponential expansion as an electric signal representation because the parameters of the exponentials have very often a physical signification. This permits also to compress information and it might be usefull to the numerical treatment. Two approaches are considered to determine the parameters of the exponential expansion: time domain and frequency domain. The determination of the equivalent complex expansion of a signal, say $y(t)$, which is a causal real valued function in the time domain, consists in identifying $y(t)$ to a sum of harmonic functions exponentially damped:

$$y(t) = \sum_{k=1}^{P/2} B_k e^{\beta_k t} \cos(\omega_k t + \theta_k), \tag{1}$$

where P is an even integer. Each of the $P/2$ elementary function or harmonic is characterized by four parameters: the amplitude B_k which is non-negative, the damping β_k which thus fulfills $\beta_k < 0$, the natural frequency ω_k and the initial phase θ_k. Expression (1) may be rewritten in a general form of an expansion of complex exponential functions, say,

$$y(t) = \sum_{k=1}^{N} A_k e^{\alpha_k t} \tag{2}$$

such that $\Re(\alpha_k) = \beta_k < 0$ and $P/2 \leq N \leq P$, via the transformation

$$\begin{cases} A_k = \frac{B_k}{2} e^{\imath\theta_k} & \alpha_k = \beta_k + \imath\omega_k \quad \text{if} \quad \omega_k \neq 0 \\ A_k = B_k \cos(\theta_k) & \alpha_k = \beta_k \quad \text{else} \end{cases} \tag{3}$$

Since $y(t)$ is a real valued function, the poles α_k and the associated residues A_k occur in complex conjugate pairs except when they occur on the negative real axis of the complex plane. In practice, the signal is recorded by a data sequence at discrete time intervals. The object is then to use all information provided by this data sequence so as to be able to extract the poles and their residues, namely to be able to refind the expected form (2) or (1). If there were no uncertainty associated with the data sequence or with the choice of the model (2) then only $2N$ separate sample times would be required to determine the poles and residues of the model exactly. In contrary, if there is uncertainty in the data sequence, say $x(t_i)$ at time t_i, it is represented by

$$x(t_i) = y(t_i) + e(t_i), \tag{4}$$

where e_i is the noise disturbance in the data. The main object now is to fit all the data $x(t_i)$ to the model $y(t_i)$ given by (2) in some best sense. The direct Prony's method or its equivalent frequency method may be used but they present some limitations (section II) and for the pratical case presented in section III a specific method has been developped [3].

II. Presentation and evaluation of Prony's method

A. Direct method based on a least square estimate

Assume that we have exact empirical values, $y_n = y(n.\Delta t)$ of $y(t)$ in (2) specified at equally spaced points, $n.\Delta t$.

$$y_n = \sum_{k=1}^{N} A_k z_k^n \qquad \forall n \in I\!N, \tag{5}$$

where $z_k = e^{\alpha_k \Delta t}$. The difficulty in solving (5) lies in the fact it is non-linear in $z_k's$. However, this difficulty can be minimized as follows. Let us consider the linear prediction polynomial

$$\Phi_N(z) = \prod_{k=1}^{N}(z - z_k^{-1}) = \sum_{k=0}^{N} a_k z^k \qquad a_0 = 1 \tag{6}$$

Hence $\Phi_N(z_k^{-1}) = 0$, $k = 1, \ldots, N$. Then the data sequence $(y_n)_{n\in\mathbb{N}}$ obey to the following recurrence

$$\sum_{k=0}^{N} a_k y_{n-k} = 0, \quad a_0 = 1, \qquad \forall n \geq N \tag{7}$$

Assume that we have now a sample of the data sequence y_n of length M, say $\{x_n, n = 1, \ldots, M\}$. The latter data sequence x_n is expected to fulfill equation (5) and necessarily it must obey to the recurrence equation (7) by taking into account the linear prediction polynomial (6). Prony method consists in three stages. The coefficients $\{a_k\}$ are determined by solving (7) in which y_n is replaced by x_n. This set of linear equations can be solved exactly if $M = 2N$ and in such case we are using the basic Prony method, or by (LSME) if $M > 2N$ and we are using the extended Prony method, say

$$\{a_k\} = -(X^T X)^{-1} X^T v, \qquad k = 1, N \tag{8}$$

where

$$X = \begin{pmatrix} x_N & x_{N-1} & x_2 & x_1 \\ x_{N+1} & x_N & x_3 & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_{M-1} & & x_{M-N+1} & x_{M-N} \end{pmatrix}, \tag{9}$$

and

$$v^T = (x_M, x_{M-1}, \ldots, x_{N+1}), \tag{10}$$

and we still call the last method by Prony method.

Solving the polynomial equation (6) finds the roots z_k, and thus the poles

$$\alpha_k = \frac{1}{\Delta t} \ln z_k \quad k = 1, N \tag{11}$$

Then (5) becomes linear in the A_k's which can be solved in the same manner as solving (8), namely by LSME,

$$A_k = (V^T V)^{-1} V^T y, \tag{12}$$

where

$$V = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \vdots & \vdots & \ldots & \vdots \\ z_1^M & z_2^M & \ldots & z_N^M \end{pmatrix}, \tag{13}$$

which is Van der Monde matrix, and

$$y^T = (y_0, y_1, \ldots, y_M). \tag{14}$$

The above procedure is namely the direct Prony method.

In the next section an alternative and more interesting method which consists in a frequency domain estimation is introduced. As it is emphasized on one example this alternative is more efficient.

B. Frequency domain analysis

Let $(y_n)_{n\in\mathbb{N}}$ be the exact empirical values of the signal $y(t)$ defined in (5). The causality of $y(t)$ permits us to assume that $y_n = 0$ for $n < 0$. By taking the inverse Fourier Transform (*IFT*), say $\hat{y}(\lambda)$, of the data sequence (y_n)

$$\hat{y}(\lambda) = \sum_{n\geq 0} y_n e^{-\imath n\lambda} \quad \forall\lambda \in [-\pi, \pi] \tag{15}$$

Using expression (5) and a direct calculation, we obtain a partial fraction expansion

$$\hat{y}(\lambda) = \sum_{k=1}^{N} \frac{A_k}{1 - z_k z}, \tag{16}$$

where $z = e^{-\imath\lambda}$ and $z_k = e^{\alpha_k \Delta t}$. Indeed, replacing y_n by it finite exponential expansion

$$y_n = \sum_{k=1}^{N} e^{\alpha_k n \Delta t} \quad \forall n \in \mathbb{N} \tag{17}$$

Inserting this in formula (15), we have

$$\begin{aligned} \hat{y}(\lambda) &= \sum_{n\geq 0}\Big(\sum_{k=1}^{N} A_k e^{\alpha_k n \Delta t}\Big) e^{-\imath n\lambda} \quad \forall\lambda \in [-\pi, \pi] \\ &= \sum_{k=1}^{N} A_k \underbrace{\Big(\sum_{n\geq 0} e^{(\alpha_k \Delta t - \imath\lambda)n}\Big)}_{I(\lambda)} \end{aligned} \tag{18}$$

It is easy to see that

$$I(\lambda) = \frac{1}{1 - e^{(\alpha_k \Delta t - \imath\lambda)}}, \tag{19}$$

and the expression (16) is proved. According to this fact we deduce that the *IFT* $\hat{y}(\lambda)$ of the data sequence (y_n) can be written as a rational function, say

$$\hat{y}(\lambda) = \frac{Q_N}{P_N}(e^{-\imath\lambda}), \tag{20}$$

where

$$P_N(z) = \prod_{k=1}^{N}(1 - z_k z) = \sum_{k=0}^{N} a_k z^k, \tag{21}$$

which is the linear prediction polynomial associated to the studied model, and

$$\begin{aligned} Q_N(z) &= \sum_{k=1}^{N} A_k \Pi_k(z) \\ &= \sum_{l=0}^{N-1} b_l z^l, \end{aligned} \quad (22)$$

where

$$\Pi_k(z) = P_N(z)/(1 - z_k z) \quad (23)$$

Let us note that $\deg(P_N) = \deg(Q_N) + 1 = N$ where N is the order of the model and all frequencies of the signal $y(t)$ are determined entirely by roots of the linear prediction polynomial P_N.

Let H be the IFT of the numerical data sequence x_n, The function H must thus be close to a rational function of the form (20) which is the IFT of the original signal, say

$$H(\lambda) = \frac{Q_N}{P_N}(e^{-\imath\lambda}) + \hat{E}(\lambda), \quad (24)$$

where $\hat{E}$ is an error function in the frequency domain. Let us note that we can refind this by taking the IFT in (4) and by using the Parseval Formula.

In order to determine the linear prediction polynomial, we consider a frequency domain error of prediction [1][2], say

$$\epsilon_N(Q_N, P_N) = \int_{-\pi}^{\pi} |P_N H - Q_N|^2 d\lambda, \quad (25)$$

The criterion (25) is quadratic and hence the conditions for ϵ_N to be minimum are given by the first order conditions [1][2]. Solving (25) determines the linear prediction polynomial P_N, and hence the poles α_k via the roots of P_N.

$$\alpha_k = \frac{1}{\Delta t}\ln(z_k). \quad (26)$$

The residues A_k are obtained by using (12).

Let us now examine the criterion (25). Introducing a frequency grid on $[0, 2\pi]$,

$$\lambda_j = \frac{2\pi j}{M} \quad \text{for} \quad j = 0, \ldots, M, \quad (27)$$

where M is supposed to be even, the criterion ϵ_N can be written in a discrete form as follows:

$$\epsilon_N(Q_N, P_N) = \sum_{k=1}^{M} |P_N(k)H(k) - Q_N(k)|^2 \quad (28)$$

According to the frequency grid (27) the data sequence $\{H(k)\}$ is exactly the $IFFT$ (Inverse Fast Fourier Transform) of the sample x_n. The coefficients (a_i) and (b_i) of the polynomials P_N and Q_N are obtained by minimizing the quadratic error ϵ_N in the frequency domain. The necessary condition for ϵ_N to be minimum is

$$\frac{\partial \epsilon_N}{\partial a_i} = 0 \; i = 0, \ldots, N \quad \frac{\partial \epsilon_N}{\partial b_i} = 0 \; i = 0, \ldots, N-1 \quad (29)$$

Hence:

$$\Re < P_N H - Q_N, H z^i >= 0 \quad i = 0, \ldots, N \; (30)$$
$$\Re < P_N H - Q_N, z^i >= 0 \quad i = 0, \ldots, N-1 \; (31)$$

where $< f, g >= \sum_{k=0}^{M-1} f(k) g^*(k)$, and $z = (e^{\frac{-2\imath\pi k}{M}})_k$. It leads to the symmetric linear system of order $2N + 1$

$$\begin{pmatrix} A & {}^tB^* \\ B & D \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad (32)$$

where $X_1 = (a_i)_{0 \le i \le N}$ and $X_2 = (b_i)_{0 \le i \le N-1}$ and the notation "*" designates the complex conjugate.

The coefficients of (32) can be expressed in a convenient form with respect to FFT (Fast Fourier Transform) computations.

$$\begin{aligned} A_{ij} &= \Re < HH^* z^{j-i}, 1 > \quad \text{for } i,j = 1, N+1 && (33) \\ B_{ij} &= -\Re < H z^{j-i+1}, 1 > \quad \text{for } i,j = 1, N && (34) \\ C_1^i &= -\Re < HH^* z^{-i}, 1 > \quad \text{for } i = 1, N+1 && (35) \\ C_2^i &= \Re < H z^{-i+1}, 1 > \quad \text{for } i = 1, N && (36) \\ D &= M I_N && (37) \end{aligned}$$

C. *Numerical results and comparison*

In this section we consider a signal with an analytical expression as a finite series of decaying exponentials. We apply Prony techniques and the frequency method for which the performance is compared.

Consider the case N=6 with the poles and residues presented in table 1 and their associated conjugates

Table 1. *analytical signal parameters*

Poles	Residues
$\alpha_1 = -0.1 + \imath 1.5$	$A_1 = 0.4 - \imath 0.2$
$\alpha_2 = -0.2 + \imath 1.8$	$A_2 = 1.0 - \imath 0.5$
$\alpha_3 = -0.3 + \imath 1.0$	$A_3 = 2.0 - \imath 1.0$

The form of this signal is illustrated in figure 1.

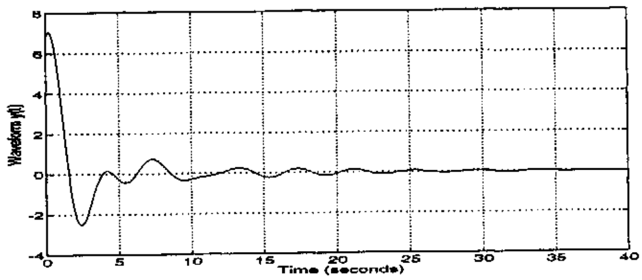

Figure 1. the waveform $y(ti)$ sampled with interval $\Delta t = 0.1$ s.

If we consider that $\Delta t = 0.25$ s, $N = 6$ and $M = 160$ which imply a time interval $[0, 40]$, we obtain the right poles and residues by the different methods. Their difference is the CPU time as shown in the following table.

Table 2. ($\Delta t = 0.25$ s)

CPU time	Direct Method	Frequency Method
(Seconds)	1.68	1.18

All routines are programmed in a Matlab environment and have been run in a SUN 10.

Consider now a time sampling $\Delta t = 0.1$ s and the same time interval, say M=400. In this case the direct Prony methods fails and provides wrong results but the frequency method gives quite acceptable results as shown in the following table.

Table 3. ($\Delta t = 0.1$ s)

	Frequency Method
Poles	$\alpha_1 = -0.1014 + \imath 1.5001$
	$\alpha_2 = -0.2000 + \imath 1.8002$
	$\alpha_3 = -0.3000 + \imath .9999$
Residues	$A_1 = 0.4050 - \imath 0.2023$
	$A_2 = 0.9979 - \imath 0.4977$
	$A_3 = 1.9978 - \imath 1.0006$

The direct Prony method as it is known is highly sensitive to the choice of the sample step. Similar conclusion can be drawn when a large time interval is considered. The above methods are limited because they depend on the time step of the signal considered and generated a great number of poles. The limitation is critical for real time simulation. Hence, in order to design a propagation model for the TNA ARENE, a new method [3], which is described bellow, has been applied to represent efficiently the modal surge impedance of cables with a limited number of poles.

III. Characteristic impedance representation

In order to design a model of cable fulfilling the real time simulation constraints, the representation of the modal characteristic impedance of such a model must be done by a minimal number of poles. In a Marti type propagation model, it is replaced by an equivalent system (figure 2), constituted by elementary impedances Z_i.

That means the characteristic impedance $Z_c(\omega)$ will be represented as follows:

$$Z_c(\omega) = k_0 + \sum_{i=1}^{N} \frac{k_i}{\jmath\omega + p_i}, \tag{38}$$

where $\{k_0, k_i, p_i\}$ are nonnegative real which are called *poles* for the p_i's and *residues* for the k_i's, respectively.

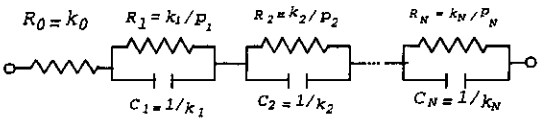

Figure 2. *R-C network approximating* $Z_c(\omega)$

In this section an iterative method permitting to obtain a representation of the modal characteristic impedance with a limited number of poles is presented. In practice, the characteristic impedance is known at some non-uniform discrete values of the frequency. In order to cover a wide frequency range with a small number of samples, a uniform logarithmically sampling is considered.

The main purpose herein is to define a minimisation criterion which permits us to determine the poles and residues independently from the time step used in the transient simulation.

Let us note that a z-transform which depends on the time step sampling can not be applied here. The only way to accomplish this is to proceed by a direct minimisation in the frequency domain.

For this purpose an adaptive criterion is constructed. At iteration of order n, n poles are determined by solving only a linear system of order $2n+1$. Only nonnegative poles and belonging to the range of desired frequencies are considered. This process is stopped when a relative error is reached.

Let N be the number of poles obtained.The next stage consists in using a re-fitting in order to determine the associated residues. This is accomplished by a least square minimisation of a quadratic criterion in the frequency domain, which the minimum is solution of a linear system of order $N+1$.

Details of this procedure are illustrated bellow.

A. Poles determination [3]

Let us consider the following series:

$$f(\omega) = k_0 + \sum_{i=1}^{N} \frac{k_i}{\jmath\omega + p_i}. \tag{39}$$

It can be rewritten as a rational function:

$$f(\omega) = \frac{Q}{P}(\jmath\omega), \tag{40}$$

where

$$P(\jmath\omega) = \prod_{i=1}^{N} (\jmath\omega + p_i) \tag{41}$$

$$Q(\jmath\omega) = k_0 P(\jmath\omega) + \sum_{i=1}^{N} k_i \pi_i(\jmath\omega)$$

and

$$\pi_i(\jmath\omega) = \prod_{k=1_{k\neq i}} (\jmath\omega + p_k) \tag{42}$$

We have $\deg(P) = \deg(Q) = N$ and

$$P(z) = \sum_{k=0}^{N} a_k z^k, \quad (43)$$

$$Q(z) = \sum_{l=0}^{N} b_l z^l, \quad (44)$$

where $z = (\jmath\omega)$. The coefficients of the polynomials P and Q are real with $a_N = 1$.

Let $(\omega_k)_{k=1,M}$ be a logarithmically sampling of the angular frequency, a first possibility for matching a rational function consists in minimising the quadratic convex frequency domain error:

$$J(P,Q) = \sum_{l=1}^{M} | < P(\omega_l) Z_c(\omega_l) - Q(\omega_l)|^2, \quad (45)$$

First of all, let us note that we are interested in frequencies varying from 0 to some kHz (10kHz). Moreover, the characteristic impedance has a real limit as the frequency becomes large. Thus, a weighted criterion which takes into account these facts may be more satisfactory.

The criterion (45) often leads to satisfactory results, but some trouble may arise because the target characteristic impedance exhibits some narrow picks and is very contrasted. Furthermore, it has only some dominant poles with a great difference. Also, the criterion (45) allows poles which values exceed ω_M, the maximal angular frequency considered. The linear system to solve may appear to be ill-conditioned or the solution to be unstable. It is then worthwhile in such a case introducing some penalization coefficients in criterion (45) as well as performing iterative improvement of the poles.

Therefore, the penalization coefficients are defined so as to put more weight with respect to weak frequencies and dominant poles.

Let ω_k be the associated sampling of the non-dimensional variables. An interesting possibility to perform iterative determination and improvement of the poles consists in minimising the successive criteria

$$J_{n+1}(P_{n+1}, Q_{n+1}) \quad (46)$$
$$= \sum_{l=1}^{M} h_{n+1}(w_l)^2 |P_{n+1}(\omega_l) Z_c(\omega_l) - Q_{n+1}(\omega_l)|^2,$$

where

$$P_0(\omega_l) = 1 \qquad \deg(P_n) \le n \quad (47)$$

and $h_{n+1}(\omega_l)$ is a weight function which depends on the form of the signal and previous poles [3]. At each iteration of order n, the sign of poles is controlled. Only nonnegative poles which values do not exceed the maximal frequency will be considered.

B. Residues calculation [3]

Let $p_1, \ldots, p_N$ be the poles obtained in the above section. Now our interest is to determine the associated residues $k_0, \ldots, k_n$. Let us consider the following functions:

$$\psi_i(\omega_l) = \frac{1}{\jmath\omega_l + p_i}, \qquad i = 1, \ldots, N \quad (48)$$

We are lead to minimising the following criterion:

$$\sum_{l=1}^{M} |Z_c(w_l) - [k_0 + \sum_{i=1}^{N} k_i \psi_i(\omega_l)]|^2, \quad (49)$$

which its minimum is solution of a symmetric linear system of order $N+1$.

C. Example

The proposed procedure was applied to the characteristic impedances associated to the single core of a Coaxial cable system. Figures 3 to 6 illustrate the results obtained. Values of poles and residues obtained are presented in tables 4 and 5. The value of residues is multiplied by 10^{-3} in table 4 and 10^{-4} in table 5 and are designated by the notation Res.

Table 4. *poles and residues for $Z_{c1}(\omega)$*

Res	0.0703	4.4746	1.4561	0.5589	0.4797
Poles		44.9908	5.8648	1.1817	0.1379

Table 5. *poles and residues for $Z_{c2}(\omega)$*

Res	0.0028	9.5755	0.1906	0.0658	0.0478
Poles		953.9659	10.5604	1.6096	0.1744

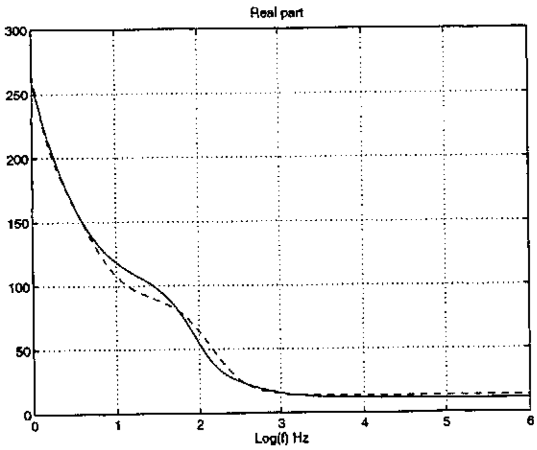

Figure 3. *real part of Z_{c1}, accurated (—) and approximated (- - -)*

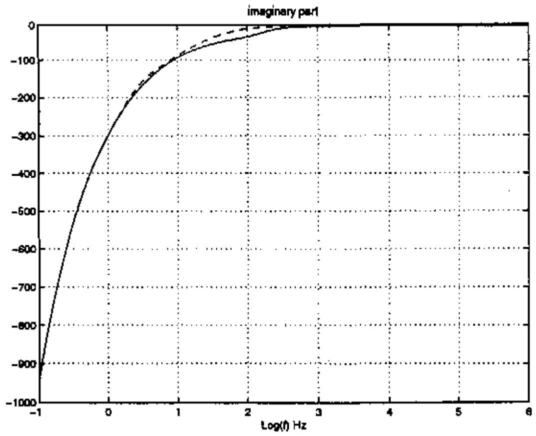

Figure 4. *imaginary part of* Z_{c1}, *accurated (—) and approximated (- - -)*

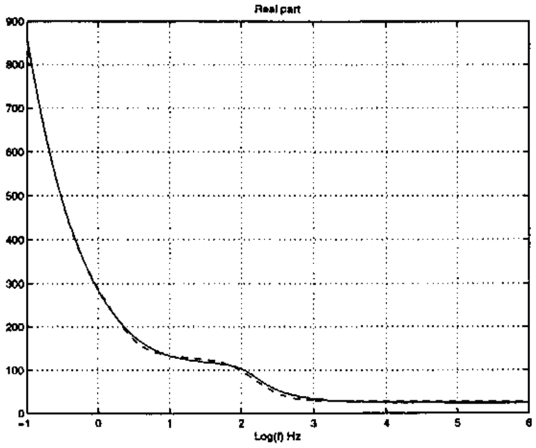

Figure 5. *real part of* Z_{c2}, *accurated (—) and approximated (- - -)*

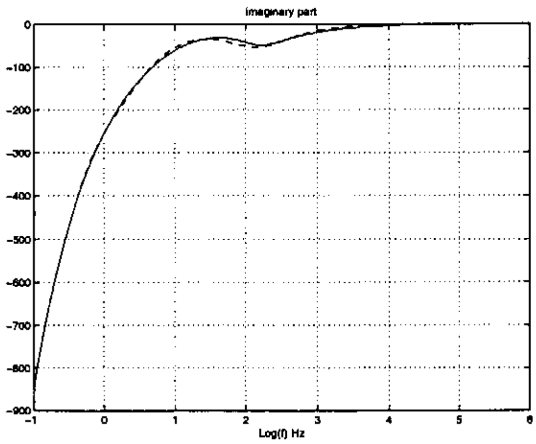

Figure 6. *imaginary part of* Z_c2, *accurated (—) and approximated (- - -)*

IV. Conclusion

Three methods for the representation of a signal by a finite exponential expansion are presented and analyzed in this paper.

The direct Prony's method is shown to be time consuming and time step sensitive.

The equivalent frequency domain method is more efficient in terms of time consuming but is time step dependent.

The last frequency method is more general, do not depend on the time step used in the transient simulation and permits the limitation of the size of the expansion. This property is of practical interest especially for real time simulation purpose.

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