Modelling of Long Grounding Conductors Using EMTP

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Abstract - **This paper addresses the problem of modelling grounding conductors whose lengths exceed 1km or more using the Electromagnetics Transients Program (EMTP). For such lengths existing techniques are commented and a new technique is presented based on the pi-equivalent circuits representation of grounding conductors. The parameters of each circuit are suitably modified to fit the impedance in the frequency domain. A suitably small integration step is chosen in order to consider the whole frequency range of the response.**

Keywords : EMTP, Grounding systems modelling, pi-equivalent circuits.

I. INTRODUCTION

For the calculation of the transient response of grounding conductors, it is necessary to use numerical analysis, as closed form solutions are not available. There have been several attempts to model grounding conductors under fast transient conditions, in the past [1-3,5-7,10-13].

By extending EMTP's capabilities the response of a grounding conductor can be computed using transmission line models[3],[13]. These models are divided in : a)division the conductor in a number of segments or b)use of a frequency dependent distributed parameter transmission line model. The advantage of EMTP is that the solution provided is fast and it is directly provided in time domain.

In this paper existing techniques for EMTP modelling of grounding the conductors are commented open and a new technique based on pi-equivalent circuits modelling is presented.

The proposed method uses division of the grounding conductor into elementary segments of varying length, each of them represented by a pi-circuit with lumped parameters. The first few meters of the conductor near the fault location which correspond to the effective length are divided into short segments increasing their length as the distance from the energisation source increases. The integration time step is suitably chosen, in order to consider the whole frequency range of the response. Results are compared to those from Inverse Fast Fourier Transform (IFFT) application.

II. THEORETICAL BACKGROUND

Series connected n pi-circuits model and equivalence to transmission line model for n→∞

a. Impedance

Grounding conductors are characterised by a series resistance R_e, a series inductance L_e, a shunt conductance G_e and a shunt capacitance C_e. They can be modelled as series connected pi-equivalent circuits with lumped R-L-C elements, where each pi-circuit corresponds to a small conductor segment (Fig.1)

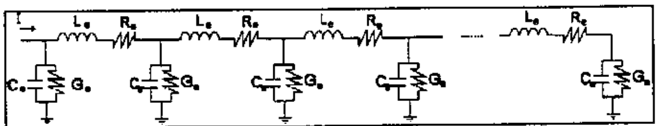


Fig. 1: Grounding conductor representation

The series and shunt impedance which correspond to each pi circuit of unit length segment are:

$Z_s = R_e + j\omega L_e$ and $Z_p = 1/(G_e + j\omega C_e)$.

A current source injecting at the end point of the conductor (Fig.1) "sees" the impedance of the conductor's n-pi circuit ladder network as:

$$Z = \frac{\alpha_1\left(b-\sqrt{Z_s^2+4Z_sZ_p}\right)^n - \alpha_2\left(b+\sqrt{Z_s^2+4Z_sZ_p}\right)^n}{\left(b+\sqrt{Z_s^2+4Z_sZ_p}\right)^n - \left(b-\sqrt{Z_s^2+4Z_sZ_p}\right)^n} \quad (1)$$

where $b = Z_s + 2Z_p$, $\alpha_1 = \dfrac{-Z_s + \sqrt{Z_s^2+4Z_sZ_p}}{2}$,

$\alpha_2 = -(Z_s + \alpha_1)$

The above formula is proved by induction and the proof is given in Appendix 1.

As n tends to infinity, the limit of Z can be evaluated:

$$\lim_{n\to\infty} Z = \lim_{n\to\infty} \frac{\alpha_1 \cdot R^n - \alpha_2}{R^n - 1} \quad (2)$$

where $$R = \frac{b-\sqrt{Z_s^2+4Z_sZ_p}}{b+\sqrt{Z_s^2+4Z_sZ_p}} \quad (3)$$

The conductor parameters for each pi-circuit R_e , L_e, G_e, C_e have to be multiplied by ℓ/n as in order to correspond to the segment length. Consequently Z_s and Z_p are redefined:

$$Z_s = \frac{R_e + j\omega L_e}{n} \cdot \ell, \quad Z_p = \frac{n}{G_e + j\omega C_e} \cdot \frac{1}{\ell} \quad (4)$$

It can be easily shown that

$$R = \left(\frac{Z_s + 2Z_p + \sqrt{Z_s^2 + 4Z_s Z_p}}{2Z_p} \right)^{-2} \quad \text{so}$$

$$\lim_{n \to \infty} R^n = \left(\frac{Z_s}{2Z_p} + 1 + \sqrt{\frac{Z_s^2}{4Z_p^2} + \frac{Z_s}{Z_p}} \right)^{-2n} = e^{-2\gamma \cdot \ell} \quad (5)$$

where $\gamma = \sqrt{(R_e + j\omega L_e)(G_e + j\omega C_e)}$ is the propagation constant. Furthermore it is:

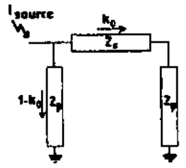
$$\lim_{n \to \infty} \alpha_1 = \lim_{n \to \infty} \sqrt{Z_s Z_p} = \sqrt{\frac{(R_e + j\omega L_e)}{(G_e + j\omega C_e)}} = Z_c,$$ which is the characteristic impedance. Consequently we have from (2)

$$\lim_{n \to \infty} Z = \frac{Z_c \cdot e^{-2\gamma\ell} + Z_c}{e^{-2\gamma\ell} - 1} = Z_c \cdot \coth(\gamma\ell) \quad (6)$$

So, as the number n of pi-circuits tends to infinity, the current at the end point of the conductor "sees" the impedance of an open ended transmission line.

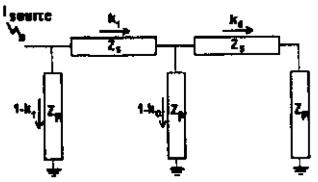
b. Current and Voltage distribution

Assuming that a current is injected at the start of the next circuit, we calculate the portion that flows along each branch, applying Kirchoff's law at the injection point:



$$k_0 \cdot (Z_s + Z_p) = (1 - k_0) \cdot Z_p \Leftrightarrow k_0 = \frac{Z_p}{Z_s + 2Z_p} \quad (7)$$

Assuming a current injected at the next 2-pi-circuits ladder network we can determine the k coefficients as before:



$$(1 - k_1) \cdot Z_p = k_1 \cdot (Z_s + (1 - k_0) \cdot Z_p) \Leftrightarrow$$

$$k_1 = \frac{Z_p}{Z_s + Z_p + (1 - k_0) \cdot Z_p} \Leftrightarrow k_1 = \frac{p + 2Z_p^2}{Z_s^2 + 4p + 3Z_p^2}$$

where k_0 is taken from (7) and $p = Z_s Z_p$

For a ladder network consisting of n series connected pi-circuits, it is:

$$k_n = \frac{R^{n+1} - 1}{r(R^{n+2} - 1)} \quad (8)$$

where R is determined from (3) and

$$r = \frac{Z_s + 2Z_p + \sqrt{Z_s^2 + 4Z_s Z_p}}{2Z_p} = R^{-1/2} \quad (9)$$

The above formula (8) can be easily proved by induction.

Considering the n-pi circuits ladder network of the grounding conductor, the current that passes through each elementary segment μ is represented by $\Pi_\mu * I_{source}$ as shown in Fig.2. The Π_μ coefficients give the current distribution along the conductor and they are accurately calculated as follows:

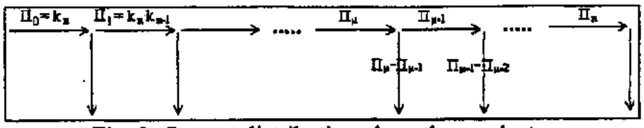


Fig. 2: Current distribution along the conductor

Applying Kirchoff's Voltage law, we have:

$$(\Pi_\mu - \Pi_{\mu+1}) \cdot Z_p = \Pi_{\mu+1} \cdot Z_s + (\Pi_{\mu+1} - \Pi_{\mu+2}) \cdot Z_p$$

or $\Pi_{\mu+2} = \frac{Z_s + 2Z_p}{Z_p} \cdot \Pi_{\mu+1} - \Pi_\mu$. It is then:

$$\Pi_\mu = \frac{r^{k+1}(R^{k+2} - 1)}{(R-1)} \Pi_{\mu-k-1} + \frac{r^k (1 - R^{k+1})}{(R-1)} \Pi_{\mu-k} \quad (10)$$

The above formula can be proved by induction. Consequently for the μ-th segment we have:

$$\Pi_\mu = \frac{r^{\mu-1}(R^\mu - 1)}{(R-1)} \Pi_1 + \frac{r^{\mu-2}(1 - R^{\mu-1})}{(R-1)} \Pi_0 \quad (11)$$

This way, the amount of the current which flows through the μ-th segment is fully determined.

The limit when n tends to infinity represents the accurate current distribution value and it is calculated as follows:

$$\lim_{n \to \infty} \Pi_\mu = \lim_{n \to \infty} \frac{r^{\mu-1}(R^\mu - 1)}{(R-1)} k_n k_{n-1} + \frac{r^{\mu-2}(1 - R^{\mu-1})}{(R-1)} k_n =$$

$$\lim_{n \to \infty} \frac{k_n \cdot r^{\mu-2}}{(R^{n+1} - 1)} (R^n - R^{\mu-1}) = \lim_{n \to \infty} r^{\mu-3} \frac{(R^n - R^{\mu-1})}{(R^{n+2} - 1)} \quad (12)$$

From (5) and (9) it is $\lim_{n \to \infty} r^n = e^{\gamma \cdot \ell}$. Also,

$$\lim_{n \to \infty} r = \lim_{n \to \infty} \left(\frac{Z_s}{2Z_p} + 1 + \sqrt{\frac{Z_s^2}{4Z_p^2} + \frac{Z_s}{Z_p}} \right) = 1$$

Consequently (12) gives:

$$\lim_{n \to \infty} r^{\mu-3} \frac{(R^n - R^{\mu-1})}{(R^{n+2} - 1)} = \lim_{n \to \infty} r^{\mu-3} \frac{(r^{-2n} - r^{-2\mu+2})}{(r^{-2n-4} - 1)}$$

and for $\mu = nx/\ell$ where ℓ is the total length of the conductor, and x the length which corresponds to μ elementary segments, we have:

$$\lim_{n \to \infty} \Pi_\mu = e^{\gamma x} \frac{e^{-2\gamma\ell} - e^{-2\gamma x}}{e^{-2\gamma\ell} - 1} = \frac{\sinh \gamma(\ell - x)}{\sinh \gamma\ell}$$

or for the current distribution along the conductor it is:

$$I(x,s) = I_{source}(s) \frac{\sinh \gamma(\ell - x)}{\sinh \gamma\ell} = I(0,s) \frac{\sinh \gamma(\ell - x)}{\sinh \gamma\ell}$$

In a similar way, the voltage distribution along the conductor can be calculated:

$$V(x,s) = I_{source}(s) \cdot Z_c \frac{\cosh \gamma(\ell - x)}{\sinh \gamma\ell}$$

The following conclusions can be drawn:

1. Current and Voltage distributions along the grounding conductor are similar as for an open-ended transmission line.
2. The pi-equivalent circuits model tends to the transmission line model, if the number of pi-circuits is increased. Error between the two models in the

frequency domain can be accurately determined using equations (1) and (11)

When the pi-equivalent circuit model is used, there is an upper limit in the maximum number of segments that can be handled by EMTP. Therefore the division of a grounding system into 0.01m or smaller segments is practically impossible. Using the lumped pi-circuits model, a computationally efficient division into 1m segments can be assumed, introducing an error within acceptable limits in the range up to 100kHz. Application into various cases has shown that when dividing the conductor into 0.01m segments, the results are accurate even in the MHz frequency range.

- **Other Pi-equivalents circuits based methods**

This method is described in [5,6] and is similar to the one described in the previous section. Each segment of the grounding conductor is represented as in fig.3, where resistance and shunt conductance are placed at the left and right end, as lumped elements.

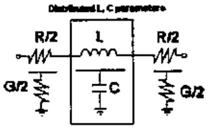


Fig. 3 Representation of a grounding conductor segment

Relations that describe the unknown node voltages and currents are based on the accurate solution of the middle part of the equivalent circuit based on the known expressions of Bergeron :

$$v_m(t-\tau)+Z_c\cdot i_m(t-\tau)=v_k(t)-Z_c\cdot i_k(t)$$

$$v_k(t-\tau)+Z_c\cdot i_k(t-\tau)=v_m(t)-Z_c\cdot i_m(t)$$

where τ=L/C is the time of travel of a wave on the line between its origin m and its end k.

Then, Kirchoff's laws, are applied, to obtain currents and voltages at the ends of the segment, including its resistance and conductance.

The numerical error introduced by this method is only due to the fact that R and G parameters are considered lumped. It is smaller than in the case of series connected pi-equivalent circuits, as the accurate solution for the lossless transmission line part of the segment is used. An example of the errors in conductor's impedance produced by this model, can be seen in fig.4 for various conductor lengths buried in various resistivity soils.

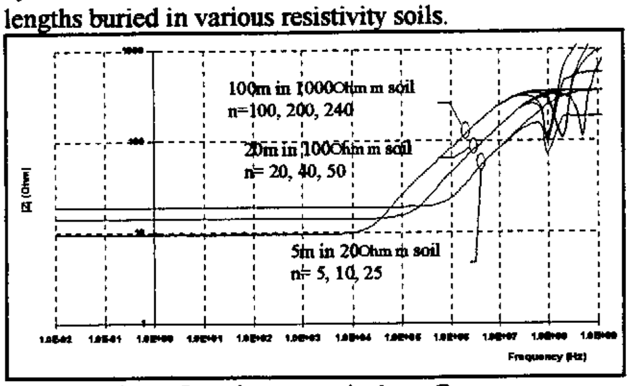


Fig. 4 : Impedance magnitude vs. Frequency

This method has the drawback that only a specific number of series connected pi-circuits gives a good model. Increasing the number of the segments and adding the series impedance the limit does not tend to the transmission line.

In order to calculate the impedance of the whole circuit that sees the current when it enters the conductor we follow the steps:

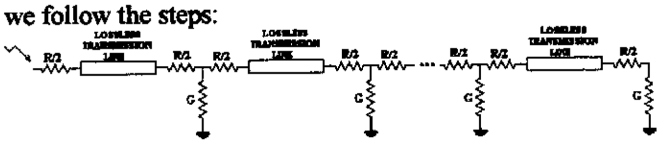


Fig. 5a

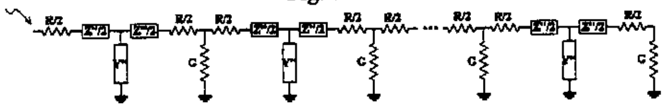


Fig. 5b

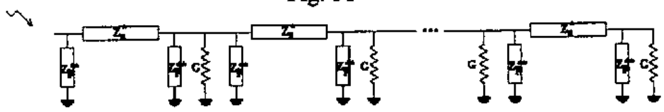


Fig. 5c

The ladder network which represents the grounding conductor is shown in Fig. 5a. In Fig. 5b the lossless transmission line part has been replaced by an equivalent T-circuit where:

$$\frac{Z^*}{2}=Z_c\frac{\cosh(\gamma\ell-1)}{\sinh(\gamma\ell)}\quad Y^*=\frac{\sinh(\gamma\ell)}{Z_c}$$

In the following Fig. 5c the T-circuit has been transformed to a pi-equivalent one. It is now possible to consider for the whole conductor a series and a shunt impedance Z_s* and Z_p* as:

$$Z_s^*=\frac{R_s\cdot\sinh(\gamma\ell)+2Z_c(\cosh(\gamma\ell)-1)}{2\sinh(\gamma\ell)}\cdot\left(2+\frac{R_s\cdot Z_c\sinh(\gamma\ell)+2Z_c^2(\cosh(\gamma\ell)-1)}{2(\sinh(\gamma\ell))^2}\right)$$

$$Z_p^*=\frac{Z_p^{**}}{2+GZ_p},\; Z_p^{**}=Y^*\left(2+\frac{R_s+Z^*}{2Y^*}\right) \qquad (13)$$

Using (1) the total impedance of the ladder network is calculated. The limit then is not the open ended transmission line impedance. So, particular attention must be paid to the number of segments that we use to model the conductor.

This method has also the same drawback as the pi-equivalent circuits method when using EMTP, ie a limited number of pi-circuits can be handled by the program.

- **Menter- J.Marti technique for long grounding conductors modelling**

According to this approach, the inherent frequency dependence of the grounding conductor characteristic impedance Zc(ω) and the corresponding propagation constant A(ω), due to the existence of resistive elements, is taken into account. The functions Zc(ω) and A(ω), the values of which are calculated using supporting routines are approximated in the frequency domain, by rational functions of the form [7,8-10]:

$$Q\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{j=1}^{m}(s+p_j)}=k_0+\sum_{j=1}^{m}\frac{k_j}{s+p_j}$$

i.e. $Z_{eq}(s) = k_0 + \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots + \frac{k_n}{s+p_n}$ (14)

where the zeros, poles and residues are denoted by zi, pj and kj respectively. The advantage of this approximation is that the left hand side of the first equation is transformed in the time domain as quickly damped exponential functions. This facilitates and accelerates the simulation calculations involving convolutions of Zc and A.

The impedance Z' and the susceptance Y' per unit length of an horizontally buried or a vertical conductor are given in [4],[10] and [11].

Frequency dependent transmission line model, has the advantage of being suitable, for a wide range of frequencies. The error introduced in this case is due to the fact that approximation of Zc(ω) and A(ω), is based on magnitude values, while phases are ignored. The phase angle of impedance given by (14) is different than that of grounding conductor's characteristic impedance. An example of this difference can be seen in figure 6 for an 1m long conductor. Fig. 6(b) is the Zc phase angle obtained by the approximation of Zc by the rational function (14). The original value is indicated by 6(a).

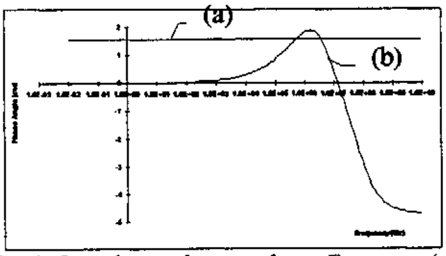


Fig. 6 : Impedance phase angle vs. Frequency(a)
Impedance approximation phase angle vs. Frequency(b)

A method is discussed in [12], for reducing the angle difference between the two approximations by adding a zero-pole pair at specific frequency. Another drawback of this method is that voltages only at the start and the end point of the conductor model can be obtained, and not at the intermediate points.

➢ **Proposed Methodology**

According to the methodology proposed in this paper, each elementary segment is represented again by a constant pi-circuit, however the length of each element is a function of its distance from the start point of the conductor.

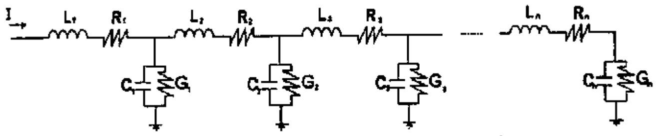


Fig. 7 Proposed conductor's model

It is proposed that only the first few meters from the energisation point of the conductor corresponding to its effective length, need to be modelled using short elementary segments. Following segments may be longer, not affecting considerably the accuracy.

Effective Length- is the length value, above which no further reduction of the conductor impedance is observed.

It can be numerically calculated as the length value where the min of the third derivative of the Z(s)-function in the log-log plane for a given frequency appears as it can be shown in the following example:

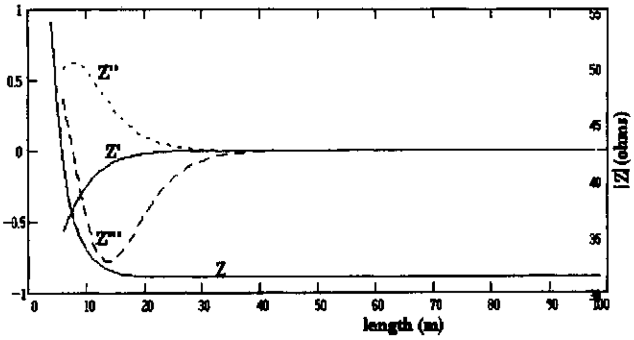


Fig. 8 : Effective length determination

Following a similar approach to the one presented in II.a, and II.b, Z(s) and I(x,s) of the proposed model are accurately calculated, and then compared to the "accurate" open ended transmission line's corresponding values. This is done in order to assure that there is an acceptable error introduced from the model in the frequency domain. An example is given for a 5.915km conductor, having pu length parameters : R_e=1.353E-6, L_e=1.185E-6, G_e=188.6599^{-1} , C_e=9.386E-12

Frequency (Hz)	|Z| of the proposed model (150 pi-circuits)	|Z| of open-ended trans. line	% Error
10^{-2}	8.079E-001	7.97E-01	1.37
10^{-1}	8.079E-001	7.97E-01	1.37
10^{0}	8.083E-001	7.98E-01	1.29
10^{1}	8.43E-001	8.32E-01	1.32
10^{2}	1.712184	1.687525	1.46
10^{3}	5.362052	5.269843	1.75
10^{4}	16.912294	16.66256	1.50
10^{5}	53.659247	52.69147	1.84
10^{6}	172.938	166.5931	3.81

It can be seen that with a suitable selection of the number of pi-circuits and adopting the effective length approach the error remains in the range of 0.01 for frequencies of 10mHz to 1MHz

The length of the μ-th segment could be exponentially related to the distance d from the injection point of the conductor.

$$length_{\mu} = A \cdot e^{B \cdot d}$$

where A and B are constants, defined by the desired accuracy.

Suitable selection of the integration time step

A time step equal to the travelling time which corresponds to the shorter conductor segment is taken in order to have results of acceptable accuracy. This way, the excitation artificially "sees" the end of the segment in time equal to the travel time.

The proposed methodology has the following advantages:

1. Computational time is reduced as only the important part of the conductor is accurately modelled. There is

no more limit in the lengths that can be handled by the program.

2. A time domain solution is calculated not only at the start and end point of the conductor, but also at any middle point.

III. APPLICATION

A 100m long grounding conductor has been firstly examined in order to show the effect of the pi-equivalents circuits reduction and the results of the proposed method are contrasted to those from application of the inverse Fast Fourier Transformation [14].

More analytically, the ladder network of the conductor has been constructed as following:

No. of Pi-circuit	Distance from start	No. of Pi-circuit	Distance from start
1÷20	1÷20 m	26	60 m
21	25 m	27	70 m
22	30 m	28	80 m
23	35 m	29	90 m
24	40 m	30	100 m
25	50 m		

Time domain results from EMTP application are shown in the following figures 9a and 9b:

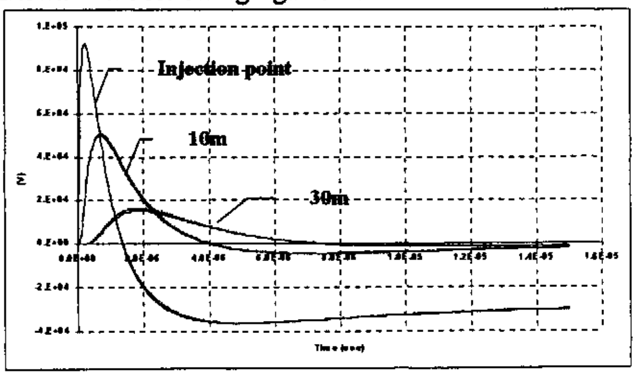


Fig. 9a: Response calculated using EMTP using a)100 1m pi-circuits and b) the simplified pi-circuits model as described above

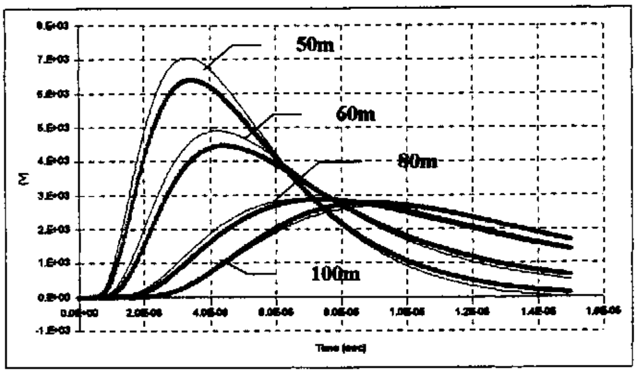


Fig. 9b: Response calculated using EMTP using a)100 1m pi-circuits(—) and b) the simplified pi-circuits model as described above(***)

IFFT solution for the same case gives for the response at the energisation point :

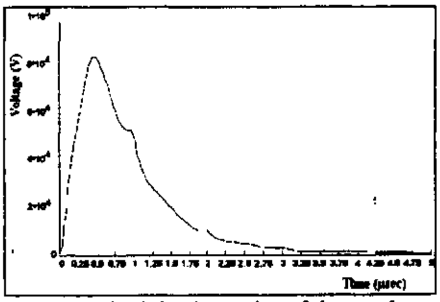


Fig. 10: Response at the injection point of the conductor using IFFT.

A conductor of 2.26 km has been examined next. It has been assumed to have the same parameters along its length. Results from the proposed method, compared to those by IFFT are given in Fig. 11. The excitation current is also plotted.

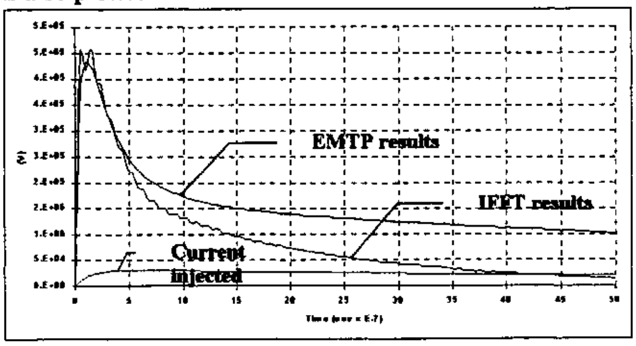


Fig. 11

A good agreement between the model proposed and IFFT implementation is observed at the results of the two cases.

IV. CONCLUSIONS

Existing techniques used for modelling grounding conductors in EMTP presented various drawbacks affecting accuracy and computational time and needed serious effort for construction of the model.

In this paper a new technique is presented for calculation of the transient response of long grounding conductors using EMTP. The model proposed is simple, easy to construct and implementation in EMTP gives results of good accuracy compared to IFFT solution.

V. AKNOWLEDGEMENTS

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Appendix 1

Neglecting the shunt branch at the start point of each circuit for simplicity it is the circuit in Fig. A.1 corresponding to n=1 and the ladder network of Fig. A.2 for n

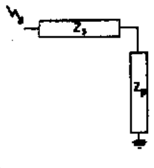

Fig. A. 1

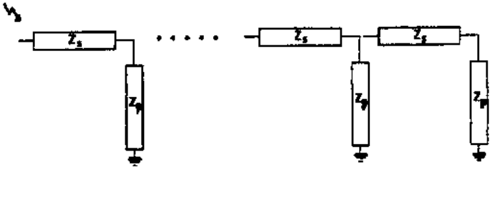

Fig. A. 2

In case of 1 circuit considered as in Fig.A.1 the impedance is $Z = Z_s + Z_p$

and the formula (1) is valid because for n=1 gives:

$$Z = \frac{\alpha_1 b - \alpha_1\sqrt{Z_s^2 + 4p} - \alpha_2 b - \alpha_2\sqrt{Z_s^2 + 4p}}{b + \sqrt{Z_s^2 + 4p} - b + \sqrt{Z_s^2 + 4p}} =$$

$$= \frac{(\alpha_1 - \alpha_2)b - (\alpha_1 + \alpha_2)\sqrt{Z_s + 4p}}{2\sqrt{Z_s^2 + 4p}} = Z_s + Z_p$$

In case of 2 circuits the impedance is:

$$Z = \frac{3Z_sZ_p + Z_s^2 + Z_p^2}{2Z_p + Z_s}$$

and the formula (1) is valid because for n=2 it gives:

$$Z = \frac{\alpha_1 \cdot \left(b - \sqrt{Z_s^2 + 4p}\right)^2 - \alpha_2 \cdot \left(b + \sqrt{Z_s^2 + 4p}\right)^2}{\left(b + \sqrt{Z_s^2 + 4p}\right)^2 - \left(b - \sqrt{Z_s^2 + 4p}\right)^2} =$$

$$= \frac{\sqrt{Z_s^2 + 4p}\left(b^2 + Z_s^2 + 4p\right) + 2bZ_s\sqrt{Z_s^2 + 4p}}{2b \cdot 2\sqrt{Z_s^2 + 4p}}$$

$$= \frac{b + Z_s^2 + 4p + 2bZ_s}{4b} = \ldots = \frac{Z_s^2 + Z_p^2 + 3p}{Z_s + 2Z_p}$$

We assume that the formula is valid for n.

In case of (n+1)-pi-circuits the impedance at the 0-start point of the ladder network is :

$$Z(n+1) = \frac{1}{\frac{1}{Z(n)} + \frac{1}{Z_p}} + Z_s = \frac{Z(n)(Z_s + Z_p) + Z_sZ_p}{Z(n) + Z_p}$$

$$= \frac{\left[\alpha_1(Z_p - \alpha_1)^n - \alpha_2(Z_p - \alpha_2)^n\right](Z_s + Z_p) + Z_sZ_p\left[-(Z_p - \alpha_1)^n + (Z_p - \alpha_2)^n\right]}{\alpha_1(Z_p - \alpha_1)^n - \alpha_2(Z_p - \alpha_2)^n + Z_p\left[-(Z_p - \alpha_1)^n + (Z_p - \alpha_2)^n\right]}$$

$$= \frac{\left[\alpha_1(Z_s + Z_p) - Z_sZ_p\right](Z_p - \alpha_1)^n - \left[\alpha_2(Z_s + Z_p) - Z_sZ_p\right](Z_p - \alpha_2)^n}{-(Z_p - \alpha_1)^{n+1} + (Z_p + \alpha_2)^{n+1}}$$

$$= \frac{\alpha_1(Z_p - \alpha_1)^{n+1} - \alpha_2(Z_p - \alpha_2)^{n+1}}{-(Z_p - \alpha_1)^{n+1} + (Z_p + \alpha_2)^{n+1}} \qquad \text{A. 1}$$

the last equality is obtained considering:

$\alpha_1(Z_s + Z_p) - Z_sZ_p = \alpha_1(Z_p - \alpha_1)$ and

$\alpha_2(Z_s + Z_p) - Z_sZ_p = \alpha_2(Z_p - \alpha_2)$

A.1 is rewritten as:

$$Z(n+1) = \frac{\alpha_1\left(b - \sqrt{Z_s^2 + 4Z_sZ_p}\right)^{n+1} - \alpha_2\left(b + \sqrt{Z_s^2 + 4Z_sZ_p}\right)^{n+1}}{\left(b + \sqrt{Z_s^2 + 4Z_sZ_p}\right)^{n+1} - \left(b - \sqrt{Z_s^2 + 4Z_sZ_p}\right)^{n+1}}$$

Consequently (1) is valid for n+1 (q.e.d.)