Equivalency of State Space Models and EMT Companion Circuit Models

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Abstract—In this paper, the relationship between state space equations and nodal analysis method is theoretically investigated and the two approaches are shown to be equivalent. The modified nodal analysis method is used to form the descriptor state space equations. These equations can be directly transferred into standard state space form. Next, the trapezoidal integration method is directly applied to these descriptor state space equations. The paper shows that the discrete descriptor state space equations, also directly yield the same admittance matrix and history current source vector as are generated when the companion circuit models in a classical electromagnetic transient (EMT) type simulation are assembled together. This proves the equivalence of the two approaches. Hence the theoretical body of knowledge from linear system analysis of the state space equations, such as eigenvalue analysis becomes applicable to the EMT simulation. A simple example is provided to show the equivalency.

Index Terms—descriptor state space equation, EMT type simulation, generalized eigenvalues

I. INTRODUCTION

Eletromagnetic transient (EMT) simulation is widely used in power system analysis [1]. There exist two main methods to simulate lumped circuit networks: state space equations and the companion circuit base nodal analysis approach introduced by Dommel [1].

For state space analysis of lumped linear electric circuits, the tree and co-tree method is used to form the standard state space equations [2]. Such a method provides a set of linearly independent state space equations of the form:

$$\dot{x} = A \cdot x + Bu \qquad (1)$$

where x and u are the state and input vector. Matrix A represents the system matrix and B denotes the input matrix.

Using a suitable integration method and suitable time-step value, the system can be simulated in the time domain. Additionally, eigenvalues of the system matrix A can yield important information about the system, such as its stability margin, time constants and oscillation modes. However, Sana and Mahseredjian [3, 4] point out a drawback of this approach in that the time and computational effort for generating state

space equations is excessive and impractical when applied to large networks. Therefore, this method is used primarily for theoretical analysis of small circuits.

On the other hand, the popular approach [1] for EMT type simulation is to convert all energy storage elements, e.g., inductors and capacitors into their companion circuit form using a suitable integration method, typically the trapezoidal rule. The companion circuit form generates a conductance in parallel with a history current source. Using nodal analysis, the admittance matrix corresponding to the connected set of companion elements can be assembled [1, 5] and the voltages in each time-step determined by solving the set of nodal equations. This method is widely used for large circuit simulations because it is straightforward to apply. However, [3] claims that the network time constants and oscillation frequencies can not directly be extracted, which is significant for stability analysis and simulation time step selection.

In the famous EMTP theory book [1], Prof. Hermann Dommel makes the comment "...*the author has never proved it, but suspects the answer* [i.e., the two methods] *are identical for an arbitrary circuit*". In this paper we prove that Prof. Dommel's suspicions are correct and that the two approaches are exactly equivalent.

Equations relating nodal equations and state space equations were presented in [6] to reconstruct the eigenvalues from the EMTP simulations. This method finds the eigenvalues in the discrete time domain in advance and maps them back to the continuous time domain according to the applied numerical algorithms. However, due to the fact that the state space equations is recovered from a EMT companion circuit, an implicit assumption is made that the nodal analysis method is theoretically identical to directly applying the trapezoidal method to state space equations, which has not been shown prior to this paper.

In this paper, descriptor state space equations (DSEs) are used to show the equivalence between the state space and companion circuit nodal analysis approaches. DSEs are formed by Modified Nodal Analysis, which usually has more state variables (which may be linearly dependent) than the standard state space equations. Chua [7, 8] has shown a generic approach for transforming the DSEs to a standard state variable form. Reference [9] also points out that the transfer function obtained by DSEs is identical to the transfer function from the standard state space, and the standard state space equations corresponding to DSEs can be consequently obtained by minimum realization of its transfer function.

In Section III, we demonstrate the equivalence of the DSEs and the EMT companion matrix. From the description in the previous paragraph, this would automatically imply the

equivalence between classical state variable equations (SVE) and the EMT approach.

This paper only considers lumped linear circuits, e.g., circuits which have resistors (R), inductors (L) and capacitors (C) and current source input. However, it is valuable to point out that other elements such as voltage source can be easily included since MNA method is used in this paper.

In the derivation, the trapezoidal integration method is applied to the DSEs, as this is the method also used in Dommel's approach. The resulting discrete time domain equations, are then represented in nodal admittance form. It is shown that the resulting admittance matrix is identical to the admittance matrix as generated in EMT solvers such as EMTP, where the admittance matric is generated from the companion models of the individual elements. A simple example is provided in Section IV to demonstrate the equivalency

Although in theory, the state variable equation model could allow for theoretical calculation of the system response, it may be impractical for a very large system. Our intention was not to propose an alternative solution method to EMT solvers, but to show the equivalency between the two methods, so that the large body of control system literature becomes available to analyze the stability and accuracy of EMT algorithms.

II. BACKGROUND

In this section, the descriptor state space equations (DSEs) can be generated based on Kirchhoff's laws using modified nodal analysis. The admittance matrix and history current source terms used by EMT type simulations are reviewed, which will be used to show the equivalence with DSEs in next section.

A. Kirchhoff's Law

In lumped circuits, the electrical behavior of the network is completely described by Kirchhoff's laws. This can be shortly summarized as below:

Assuming branch currents $i_1 \cdots i_l$ entering any node n in the circuit, Kirchhoff's current law (KCL) gives $\sum_{k=1}^{l} i_{nk} = 0$. Similarly, for a loop p composed of m branches with voltages $v_1 \cdots v_m$, Kirchhoff's voltage law (KVL) states that $\sum_{i=1}^{m} v_{pi} = 0$.

For a practical circuit, there are usually many loops and nodes. In order to form the standard state space equations with a non-singular system matrix, the circuit is partitioned into a normal tree and normal co-tree methods to systematically generate the state space equations [2]. However, Sana and Mahseredjian [3, 4] state that this method is impractical for very large circuits because of the excessive computation time to formulate the state space. As an alternative, the MNA method, which is widely used in computer aided design [10] can be used. The MNA equations can be easily generated from the circuit netlist but in the form of descriptor state space equations, which are essentially Differential-algebraic equations (DAEs) and the descriptor state variables may be linear dependent.

B. Description of the Circuit by a Node Incidence Matrix

Consider a circuit with n nodes (excluding the datum or ground node) and b branches. We define an orientation for each branch, i.e. one node is set to be the "starting" node while the other is the "end". The node incidence matrix with elements as

shown below relates the nodes to the branches [2]. Note that this matrix can be directly formed based on the circuit netlist, which is the same information used for EMT type simulations:

$$a_{ij} = \begin{cases} 1, & if\ branch\ j\ has\ start\ node\ i \\ -1, & if\ branch\ j\ has\ end\ node\ i \\ 0, & otherwise \end{cases} \quad (2)$$

Consequently, collecting all branch currents into one vector:
$$\boldsymbol{i} = [i_1 \quad \cdots \quad i_b]^T$$
The KCL equations are directly represented as below:
$$\boldsymbol{A} \cdot \boldsymbol{i} = 0 \quad (3)$$
where $\boldsymbol{A}$ is the incidence matrix:
$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1b} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nb} \end{pmatrix}$$

Using vector $\boldsymbol{e}$ and $\boldsymbol{v}$ to represent node voltage and branch voltage, the node incidence matrix also represents the mapping from node to branch voltage as below:
$$\boldsymbol{v} = \boldsymbol{A}^T \cdot \boldsymbol{e} \quad (4)$$

C. Generation of Descriptor State Space Equations

Assume there are n_C capacitors, n_L inductors, n_R resistors and n_I current sources. Ordering all the columns in matrix $\boldsymbol{A}$ in the sequence of capacitor, inductor, resistor, current source branches, i.e., $\boldsymbol{A} = (\boldsymbol{A_C}, \boldsymbol{A_L}, \boldsymbol{A_R}, \boldsymbol{A_I})$, equation (4) can be rewritten as below:

$$\begin{pmatrix} \boldsymbol{v_c} \\ \boldsymbol{v_L} \\ \boldsymbol{v_R} \\ \boldsymbol{v_I} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A_c}^T \\ \boldsymbol{A_L}^T \\ \boldsymbol{A_R}^T \\ \boldsymbol{A_I}^T \end{pmatrix} \cdot \boldsymbol{e} \quad (5)$$

Where $\boldsymbol{v_c}, \boldsymbol{v_L}, \boldsymbol{v_R}, \boldsymbol{v_I}$ denote capacitor, inductor, resistor and current source branch voltage vector respectively.

Consequently, substituting matrix $\boldsymbol{A} = (\boldsymbol{A_C}, \boldsymbol{A_L}, \boldsymbol{A_R}, \boldsymbol{A_I})$ to equation (3) gives:
$$\boldsymbol{A_c} \cdot \boldsymbol{i_c} + \boldsymbol{A_L} \cdot \boldsymbol{i_L} + \boldsymbol{A_R} \cdot \boldsymbol{i_R} + \boldsymbol{A_I} \cdot \boldsymbol{i_I} = \boldsymbol{0} \quad (6)$$
Substituting the RLC elements relation and equation (5) to (6) gives:

$$\begin{pmatrix} \boldsymbol{A_c} \cdot \boldsymbol{C} \cdot \boldsymbol{A_c}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{i_L} \end{pmatrix}$$
$$= -\begin{pmatrix} \boldsymbol{A_R} \cdot \boldsymbol{G} \cdot \boldsymbol{A_R}^T & \boldsymbol{A_L} \\ -\boldsymbol{A_L}^T & \boldsymbol{0} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{i_L} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A_I} \\ \boldsymbol{0} \end{pmatrix} \cdot \boldsymbol{i_I} \quad (7)$$

where $\boldsymbol{C}$, $\boldsymbol{G}$, and $\boldsymbol{L}$ are diagonal matrixes containing all the capacitance, admittance and inductance values respectively. With $\boldsymbol{C'} = \boldsymbol{A_c} \cdot \boldsymbol{C} \cdot \boldsymbol{A_c}^T$ and $\boldsymbol{G'} = \boldsymbol{A_R} \cdot \boldsymbol{G} \cdot \boldsymbol{A_R}^T$ (6) becomes (7):

$$\begin{pmatrix} \boldsymbol{C'} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{i_L} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{G'} & \boldsymbol{A_L} \\ -\boldsymbol{A_L}^T & \boldsymbol{0} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{i_L} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A_I} \\ \boldsymbol{0} \end{pmatrix} \cdot \boldsymbol{i_I} \quad (8)$$

which is in the form:
$$\boldsymbol{E} \cdot \dot{\boldsymbol{x}} = -\boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \quad (9)$$
Equation (9) is known as descriptor state space equation [11].

In standard state space equations of the form (1) the state variables are the linearly independent capacitor branch voltages and linearly independent inductor branch currents (if inductor cut-sets or capacitor loops exists, not all inductor currents and capacitor voltages are SVs). These are a minimal set and are

linearly independent. In DSEs, all node voltages e and inductor branch currents are the descriptor state variables (might be linear dependent), which are a linear combination of the standard state variables. Consequently, the matrix E may be singular and thus non-invertible. If it was invertible, (9) could directly be transformed to (1) by inverting E. Reference [7, 8] describe methods to transform (9) to standard state variable, and eigenvalues of (1) are the same as the generalized eigenvalues [12] of (9). As a results, (9) and (1) represent the same physical system and embody the same information.

D. Update Equation in EMT Type Simulation

The EMT algorithm is a systematic approach to obtaining update values for node voltages based on the input current sources and past history of inductor and capacitor currents and voltages. All inductor and capacitor branches are first discretized into a companion circuit consisting of a conductance and a history current source as shown in Fig. 1 [1].
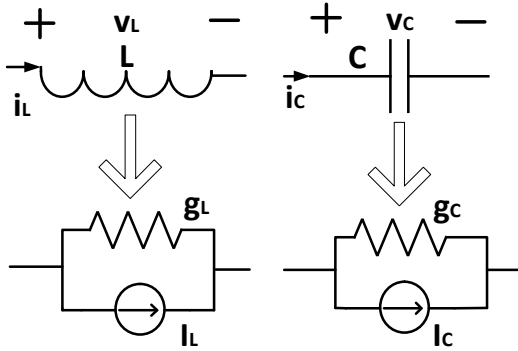
Fig. 1. Companion circuit model for inductor and capacitor

For the jth inductor or kth capacitor

$$i^{(k)}{}_c(t) = g_c{}^{(k)} \cdot v^{(k)}{}_c(t) + I^{(k)}{}_c(t - \Delta t)$$
$$i_L{}^{(j)}(t) = g_L{}^{(j)} \cdot v^{(j)}{}_L(t) + I^{(j)}{}_L(t - \Delta t) \quad (10)$$

The entire set of companion circuit branch equations (with n_C capacitors and n_L inductors) then become as in:

$$\begin{pmatrix} i^{(1)}{}_c(t) \\ \vdots \\ i^{(n_C)}{}_c(t) \end{pmatrix} = \begin{pmatrix} g_c{}^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_c{}^{(n_C)} \end{pmatrix} \cdot \begin{pmatrix} v^{(1)}{}_c(t) \\ \vdots \\ v^{(n_C)}{}_c(t) \end{pmatrix}$$
$$+ \begin{pmatrix} I^{(1)}{}_c(t - \Delta t) \\ \vdots \\ I^{(n_C)}{}_c(t - \Delta t) \end{pmatrix}$$

And

$$\begin{pmatrix} i^{(1)}{}_L(t) \\ \vdots \\ i^{(n_L)}{}_L(t) \end{pmatrix} = \begin{pmatrix} g_L{}^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_L{}^{(n_L)} \end{pmatrix} \cdot \begin{pmatrix} v^{(1)}{}_L(t) \\ \vdots \\ v^{(n_L)}{}_L(t) \end{pmatrix}$$
$$+ \begin{pmatrix} I^{(1)}{}_L(t - \Delta t) \\ \vdots \\ I^{(n_L)}{}_L(t - \Delta t) \end{pmatrix}$$

Which can be written in matrix and vector form as below:

$$i_c(t) = g_c \cdot v_c(t) + I_c(t - \Delta t)$$
$$i_L(t) = g_L \cdot v_L(t) + I_L(t - \Delta t) \quad (11)$$

Where g_c and g_L are diagonal matrixes containing all the equivalent inductor and capacitor admittance respectively, i_c, v_c, i_L, v_L are column vectors which represent capacitor and inductor branch currents and voltages. Their elements are

arranged corresponding to matrix g_c and g_L. I_c, I_L are history current source vectors:

$$I_L(t - \Delta t) = i_L(t - \Delta t) + g_L \cdot v_L(t - \Delta t)$$
$$I_C(t - \Delta t) = -i_C(t - \Delta t) - g_c \cdot v_c(t - \Delta t) \quad (12)$$

Consequently, nodal analysis is applied by substituting equation (11) and (12) into (5) and (6):

$$\left[A_c \cdot g_c \cdot A_c{}^T + A_L \cdot g_L \cdot A_L{}^T + A_R \cdot G \cdot A_R{}^T \right] \cdot e(t)$$
$$= -A_I \cdot i_I(t) - A_L \cdot I_L(t - \Delta t) - A_c \cdot I_C(t - \Delta t) \quad (13)$$

Simple algebraic manipulation gives:

$$\left[\frac{2C'}{\Delta t} + A_L \cdot \frac{\Delta t}{2} \cdot L^{-1} \cdot A_L{}^T + G' \right] \cdot e(t)$$
$$= -A_I \cdot i_I(t) - A_L \cdot I_L(t - \Delta t) - A_c \cdot I_C(t - \Delta t) \quad (14)$$

Equation (14) will be used in next section to show the equivalency with the DSE approach.

In EMT type simulation field, Equation (14) is usually known as new node voltage update equations form as below in EMT type simulations:

$$Y_{emt} \cdot e(t) = I_{his}(t - \Delta t) + I_s(t) \quad (15)$$

where:

$$I_{his}(t - \Delta t) = -A_L \cdot I_L(t - \Delta t) - A_c \cdot I_C(t - \Delta t)$$
$$I_s(t) = -A_I \cdot i_I(t)$$

III. RELATIONSHIP BETWEEN DESCRIPTOR STATE SPACE EQUATIONS AND EMT TYPE SIMULATIONS

This section derives the update equation for new node voltages based on the descriptor state space and shows that it is identical to the update equation (14) for EMT type companion matrix.

A. Update Equation Derived from Descriptor State Equations

Directly applying the trapezoidal method to equations (9) gives:

$$E \cdot x(t) = E \cdot x(t - \Delta t) + \frac{-A \cdot \Delta t}{2} \cdot \left(x(t) + x(t - \Delta t) \right) + \frac{B \cdot \Delta t}{2} \left(u(t) + u(t - \Delta t) \right) \quad (16)$$

Substituting equation (8) gives:

$$\begin{pmatrix} C' & 0 \\ 0 & L \end{pmatrix} \cdot \begin{pmatrix} e(t) \\ i_L(t) \end{pmatrix} = \begin{pmatrix} C' & 0 \\ 0 & L \end{pmatrix} \cdot \begin{pmatrix} e(t - \Delta t) \\ i_L(t - \Delta t) \end{pmatrix} + \frac{\Delta t}{2} \cdot - \begin{pmatrix} G' & A_L \\ -A_L{}^T & 0 \end{pmatrix} \cdot \left(\begin{pmatrix} e(t) \\ i_L(t) \end{pmatrix} + \begin{pmatrix} e(t - \Delta t) \\ i_L(t - \Delta t) \end{pmatrix} \right)$$
$$+ \frac{\Delta t}{2} \cdot - \begin{pmatrix} A_I \\ 0 \end{pmatrix} \cdot (i_I(t - \Delta t) + i_I(t)) \quad (17)$$

B. First Row Expansion

Simplifying first row of equation (17) gives:

$$\frac{2C'}{\Delta t} \cdot e(t) = \frac{2C'}{\Delta t} \cdot e(t - \Delta t) - (G' \quad A_L) \cdot \left(\begin{pmatrix} e(t - \Delta t) \\ i_L(t - \Delta t) \end{pmatrix} + \begin{pmatrix} e(t) \\ i_L(t) \end{pmatrix} \right) + I_s \quad (18)$$

where $I_s = -\begin{pmatrix} A_I \\ 0 \end{pmatrix} \cdot (i_I(t) + i_I(t - \Delta t))$.

Using KCL, we have $-(G' \quad A_L) \cdot \begin{pmatrix} e(t - \Delta t) \\ i_L(t - \Delta t) \end{pmatrix} = A_c \cdot$ $i_c(t - \Delta t) + A_I \cdot i_I(t - \Delta t)$. Therefore:

$$\frac{2C'}{\Delta t} \cdot e(t) = -(G' \quad A_L) \cdot \left(\begin{pmatrix} e(t) \\ i_L(t) \end{pmatrix} \right) + (\frac{2C'}{\Delta t} \cdot e(t -$$
$$\Delta t) + A_c \cdot i_c(t - \Delta t) - A_I \cdot i_I(t)) \quad (19)$$

Since:

$$\frac{2C'}{\Delta t} \cdot e(t - \Delta t) = A_c \cdot g_c \cdot A_c{}^T \cdot e(t - \Delta t)$$
$$= A_c \cdot g_c \cdot v_C(t - \Delta t) \quad (20)$$

Expanding equation (19) gives:

$$\left(\frac{2C'}{\Delta t} + G' \right) \cdot e(t)$$
$$= -A_L \cdot i_L(t) + A_c \cdot g_c \cdot v_C(t - \Delta t) + A_c \cdot i_c(t - \Delta t) - A_I \cdot i_I(t)$$
$$= -A_L \cdot i_L(t) - A_I \cdot i_I(t) - A_c \cdot I_C(t - \Delta t) \quad (21)$$

C. Second Row Expansion

Similarly, expanding and simplifying the second row of equation (17) gives:

$$i_L(t) = i_L(t - \Delta t) - \frac{\Delta t}{2} \cdot (-L^{-1} \cdot A_L{}^T \quad 0) \cdot$$
$$\left(\begin{pmatrix} e(t - \Delta t) \\ i_L(t - \Delta t) \end{pmatrix} + \begin{pmatrix} e(t) \\ i_L(t) \end{pmatrix} \right) \quad (22)$$

where the bold **0** represents zero matrix.

Substituting equation (12) with simple algebra gives:

$$i_L(t) = I_L(t - \Delta t) + \frac{\Delta t}{2} \cdot L^{-1} \cdot A_L{}^T \cdot e(t) \quad (23)$$

D. Update equation for node voltages derived from Descriptor State Space Equations

In direct numerical integration of DSEs, the descriptor state variables include all node voltage and inductor currents. On the other hand, EMT simulation approaches using companion circuits and nodal analysis, only update node voltages. Hence to obtain the node voltage updates from DSEs, we need to eliminate inductor current term. This can be done by substituting equation (23) to the first row expansion (21) as below:

$$\left(\frac{2C'}{\Delta t} + G' + A_L \cdot \frac{\Delta t}{2} \cdot L^{-1} \cdot A_L{}^T \right) \cdot e(t)$$
$$= -A_I \cdot i_I(t) - A_L \cdot I_L(t - \Delta t) - A_c \cdot I_C(t - \Delta t)$$
Which is also in the form of: $\quad (24)$
$$Y_{emt} \cdot e(t) = I_{his}(t - \Delta t) + I_s(t)$$

Equation (14) used for EMT type simulations, and Equation (24) obtained by the integration of the DSEs are exactly the same, which completes the proof of equivalency.
Q.E.D.

Although DSEs rather than classical state variable equations (SVEs), were used as the system equations, the equivalence to SVEs immediately follows from earlier work [7-9] which has shown that any set DSEs can be transformed to a set of SVEs.

IV. DEMONSTRATION EXAMPLE

In this section, a simple example is provided that shows the resulting update equations for Dommel's method and DSE are the same. Also simulations for two different time steps using the two approaches are shown and are identical.
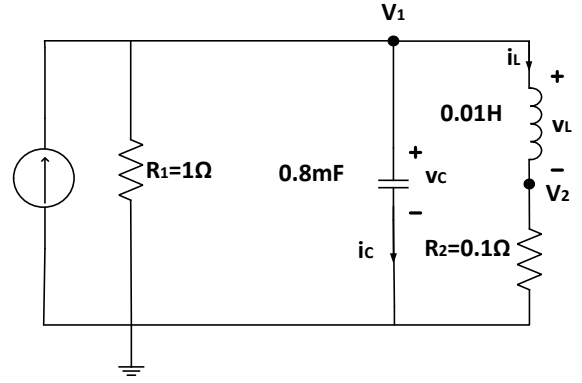

Fig. 2. Example 1

The descriptor state space equations from MNA method can be formed as below:
From the circuit netlist, we have:

$$\begin{pmatrix} v_c \\ v_L \\ v_{R1} \\ v_{R2} \\ v_I \end{pmatrix} = \begin{pmatrix} A_c{}^T \\ A_L{}^T \\ A_{R1}{}^T \\ A_{R2}{}^T \\ A_I{}^T \end{pmatrix} \cdot e = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, C = 0.0008, L = 0.01$$

Substituting equation (7) gives the descriptor state space equations:

$$\begin{pmatrix} 0.0008 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.01 \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} V_1 \\ V_2 \\ i_L \end{pmatrix}$$
$$= - \begin{pmatrix} 1 & 0 & 1 \\ 0 & 10 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ i_L \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \cdot (i_s) \quad (25)$$

Directly substituting the parameters to equation (24) gives the admittance matrix as below:

$$Y_{emt} = \frac{2C'}{\Delta t} + G' + A_L \cdot \frac{\Delta t}{2} \cdot L^{-1} \cdot A_L{}^T$$
$$= \begin{pmatrix} \frac{0.0016}{\Delta t} + 1 + 50 \cdot \Delta t & -50 \cdot \Delta t \\ -50 \cdot \Delta t & 10 + 50 \cdot \Delta t \end{pmatrix} \quad (26)$$

With a history current source term:

$$I_{his}(t - \Delta t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot I_L(t - \Delta t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot I_C(t - \Delta t) \quad (27)$$

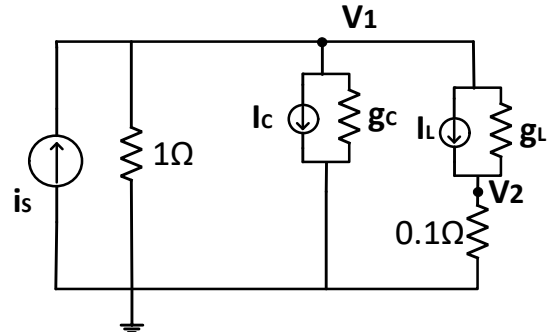The corresponding companion circuit formed from Dommel's method is shown below in Fig. 3:

Fig. 3: Companion Circuit for Example 1

In Fig. 3, $g_C = \frac{0.0016}{\Delta t} s$ and $g_L = 50 \cdot \Delta t \ s$ are the equivalent admittance term. The history current source term $I_C(t - \Delta t)$ and $I_L(t - \Delta t)$ are:

$$I_C(t - \Delta t) = -\frac{0.0016}{\Delta t} \cdot v_c(t - \Delta t) - i_c(t - \Delta t)$$

$$I_L(t - \Delta t) = i_L(t - \Delta t) + 50 \cdot \Delta t \cdot v_L(t - \Delta t) \quad (28)$$

Assembling the nodal admittance matrix and history current vector from Fig.3 gives:

$$\boldsymbol{Y}_{emt} = \begin{pmatrix} \frac{0.0016}{\Delta t} + 1 + 50 \cdot \Delta t & -50 \cdot \Delta t \\ -50 \cdot \Delta t & 10 + 50 \cdot \Delta t \end{pmatrix} \quad (29)$$

And

$$\boldsymbol{I}_{his}(\text{t} - \Delta t) = \begin{pmatrix} -I_L(t - \Delta t) - I_C(t - \Delta t) \\ I_L(t - \Delta t) \end{pmatrix} \quad (30)$$

which is identical to equation (26) and (27), as it should be, because the DSE approach and Companion Circuit approach were proved to be identical in Section III.
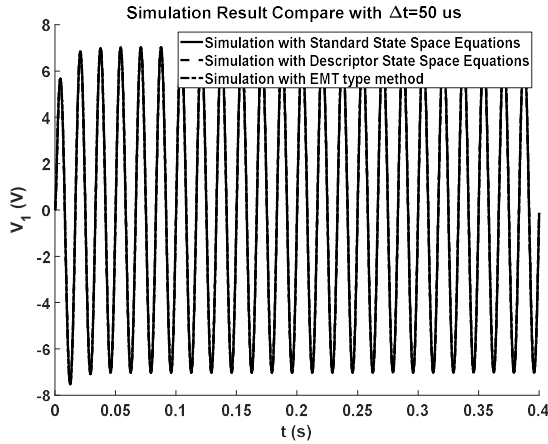

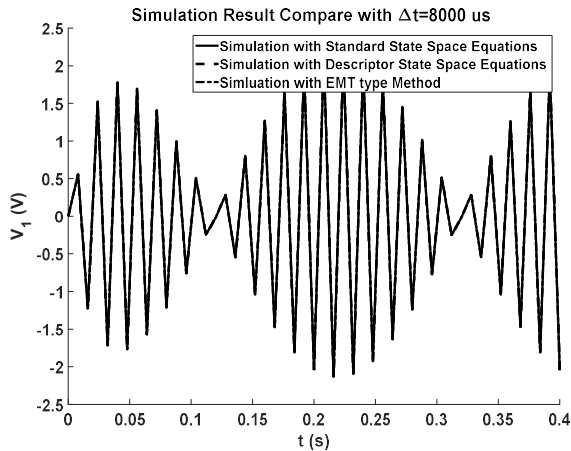Fig 4. Node 1 voltage for $\Delta t = 50 \ \mu s$

Fig 5. Node 1 voltage for $\Delta t = 8000 \ \mu s$

Fig. 4. shows the voltage of node 1 for a time-step of 50 μs, using Dommel's method, as well as from integration of the DSEs and SVEs by Trapezoidal method. Similarly, Fig. 5. shows the same for a time-step of 8000 μs. In both cases the results are exactly overlapping, as they should be due to the exact equivalence. Note the 8000 μs timestep is too large for

accuracy and gives useless results, however, the point here was to show that regardless of the time-step used, the two equivalent approaches yield the same results, accurate or otherwise!

V. CONCLUSION

In this paper, transient simulation based on state variable approach is theoretically shown to be equivalent to Dommel's companion circuit/nodal analysis approach.

To show the equivalence, Dommel's method is first applied and the update equation consisting of the overall resultant admittance matrix and history terms is derived. Then Descriptor State Space Equations (DSEs) which can be directly obtained from the MNA method are obtained. By algebraic manipulation of the DSEs, the resulting update equations for node voltages are derived and have the form of an admittance matrix and history terms. The admittance matrix and history terms obtained from the DSEs are exactly the same, thereby proving the equivalence. Although DSEs rather than classical state variable equations (SVE), were used as the DSEs equations are easily obtainable for arbitrary circuits, the equivalence to SVEs immediately follows from earlier work [7-9]which has shown that DSEs can be transformed to SVEs.

One simple example is provided in the end to demonstrate the equivalency. Simulations with two different time-steps are presented and show that the simulations too are exactly the same.

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