The Efficient Simulation of Multiple Time Scale Systems

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Abstract— **Integration methods are analyzed permitting the efficient calculation of electromagnetic and electromechanical transients in power systems using the same network model. The described methods apply commonly used approximations only by increasing integration step size. They are valid for balanced and unbalanced network conditions. An example covering a wide range of applications demonstrates the efficiency of the implemented algorithm.**

I. Introduction

Transients in power systems are characterized by eigenvalues which are in different order of magnitude. At least two groups of eigenvalues can be identified:

- Large eigenvalues, electromagnetic transients.
- Small eigenvalues, electromechanical transients.

For each group of transients, different approximations can be applied. Some programs offer the possibility to switch between a complete dynamic network representation for electromagnetic transients and a steady state network representation for electromechanical transients.

Particularly under unbalanced network conditions, this sudden change of the network model causes a parasitic excitation of the whole system.

In this paper, integration methods are presented which can be used for the whole range of transients in power systems using only one network model.

By increasing integration step size, the network model transits smoothly from a dynamic representation appropriate for electromagnetic transients to a representation describing the steady network state correctly.

Stability and precision of these integration methods and the control of integration step size will be analyzed. An example visualizes how the presented methods work in DIgSILENT [1], a standard program for power system analysis.

Because of the increasing importance of FACTS and other nonlinear devices, the described algorithm for the linear part of the electrical grid has to be seen in combination with a multiple step size algorithm employing multiple time scale properties in nonlinear parts of the electrical network.

[1]Digital Simulation of Electrical Networks

II. Multiple Time Scale Systems

The eigenvalues of multiple time scale systems are in different order of magnitude. State variables should be chosen in such a manner that a big number of them is not influenced by large eigenvalues.

In this paper, the electrical power system is described by a two time scale system. All state variables related to the electrical grid, like line currents, transformer currents, voltages across capacitances and also stator currents of electrical machines belong to the subsystem with large eigenvalues. These state variables and the according equations will therefore be called *fast subsystem.*

All mechanical variables and state variables related to the rotor of electrical machines are only weakly influenced by large eigenvalues and are therefore assigned to the *slow subsystem.*

A. Multiple Time Scale System of Second Order

In order to derive some fundamental properties of multiple time scale systems, a multiple time scale system of second order is analyzed in this section. Consider the following set of linear differential equations with constant coefficients:

$$\begin{pmatrix} T_s\dot{x}_s \\ T_f\dot{x}_f \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} \tag{1}$$

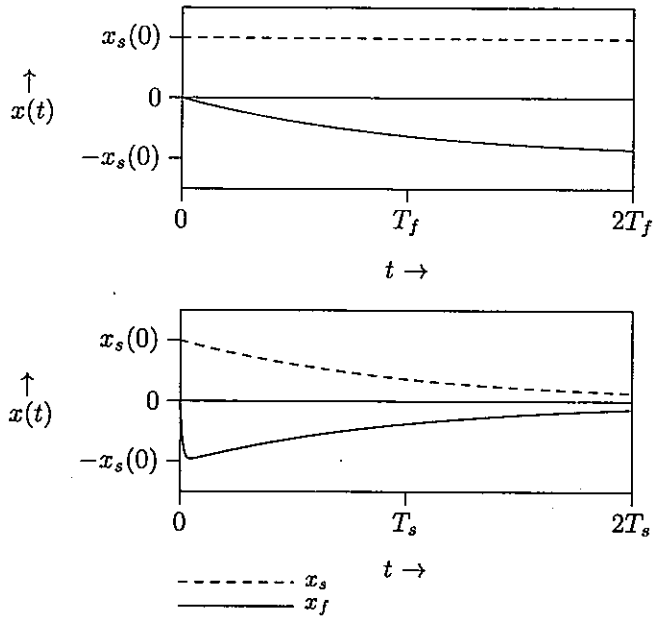
With $T_s \gg T_f$ and all coefficients a_{ij} being of the same order of magnitude, (1) is a multiple time scale system.

The relation $T_s \gg T_f$ can be expressed by introducing the *singular perturbation* factor ε [1] and writing $T_f = \varepsilon T_s$ with $\varepsilon \ll 1$.

B. Singular Perturbation Analysis

Approximate values for λ_1 and λ_2 can be obtained by transforming the set of differential equations (1) in different time scales and analyzing fast and slow transient separately (see Fig.1).

The time transformation $t = \varepsilon\hat{t}$ zooms in time scale and is therefore appropriate for analyzing the fast transient.



($\varepsilon = 0.01$, $a_{11} = a_{21} = a_{22} = -1$, $a_{12} = 0$)

Fig. 1. State variables in different time scales.

In the new time scale, (1) can be expressed by:

$$\begin{aligned} T_s \frac{dx_s}{d\hat{t}} &= \varepsilon a_{11} x_s + \varepsilon a_{12} x_f \\ \varepsilon T_s \frac{dx_f}{d\hat{t}} &= \varepsilon a_{21} x_s + \varepsilon a_{22} x_f \end{aligned} \tag{2}$$

With $\varepsilon \ll 1$, equation (2) can be approximated for small time scale analysis:

$$\begin{aligned} \frac{dx_s}{d\hat{t}} &\approx 0 \\ T_s \frac{dx_f}{d\hat{t}} &= a_{21} x_s + a_{22} x_f \end{aligned} \tag{3}$$

The corresponding eigenvalue is:

$$\lambda_1 = \frac{\hat{\lambda}_1}{\varepsilon} = \frac{a_{22}}{\varepsilon T_s} = \frac{a_{22}}{T_f} \tag{4}$$

For analyzing the slow subsystem, time scale has to be zoomed out by $t = \check{t}/\varepsilon$.

Transforming (1) into the large time scale leads to:

$$\begin{aligned} T_f \frac{dx_s}{d\check{t}} &= a_{11} x_s + a_{12} x_f \\ \varepsilon T_f \frac{dx_f}{d\check{t}} &= a_{21} x_s + a_{22} x_f \end{aligned} \tag{5}$$

for $\varepsilon \rightarrow 0$, the large time scale approximation is obtained with:

$$0 \approx a_{21} x_s + a_{22} x_f \tag{6}$$

Hence, the eigenvalue of the slow subsystem is:

$$\lambda_2 = \varepsilon \check{\lambda}_2 = \frac{a_{11} a_{22} - a_{21} a_{22}}{a_{22} T_s} \tag{7}$$

C. Time Dependent Coefficient Matrix

Approximation (6) is only valid if the system-matrix is linear and time independent.

Only balanced networks can be described by such a type of system matrix. In unbalanced network states, coupling elements between fast and small subsystem are always sinusoidal functions related to machine rotation. Steady state of x_f is therefore also sinusoidal. With

$$\begin{aligned} x_{f_{steady}}(\check{t}) &= \sum_{k=-\infty}^{\infty} \underline{X}_k e^{j\omega_k \frac{\check{t}}{\varepsilon}} \\ \frac{dx_{f_{steady}}}{d\check{t}} &= \sum_{k=-\infty}^{\infty} j\frac{\omega_k}{\varepsilon} \underline{X}_k e^{j\omega_k \frac{\check{t}}{\varepsilon}} \end{aligned} \tag{8}$$

the left side of the second equation in (5) approximates only zero for frequencies which are low with respect to $1/T_f$.

Therefore, in case of unbalanced networks, large time scale approximation has to be modified in the following way:

$$\dot{x}_f \approx \dot{x}_{f_{steady}} \tag{9}$$

III. Numerical Solution

In order to digitally simulate a continuous time system, it has to be transformed into a discrete time system. This can be done by replacing each integrator by a discrete time function, oftenly called *numerical integration method.* All variables are then analyzed at discrete points $t + kh$ (k: integer value, h: integration step size).

In small time scale analysis, when transients of the fast subsystem are of interest, the numerical integration method must map the eigenvalues of the original continuous time system closely to the eigenvalues of the derived discrete time system. Here, an efficient way to simulate multiple time scale systems is the use of an individual integration step size h in each subsystem.

In addition, the simulation algorithm should be designed in such a manner that it profits by the multiple time scale approximations described in the previous section.

Fig. 2 demonstrates the principles of a multirate algorithm for a two time scale system using two different integration step sizes h_f and h_s for the fast and the slow subsystem respectively. A detailed analysis of a multirate algorithm can be found in [2].

After fast transients have faded away, large time scale approximations can be applied. Consequently, only the

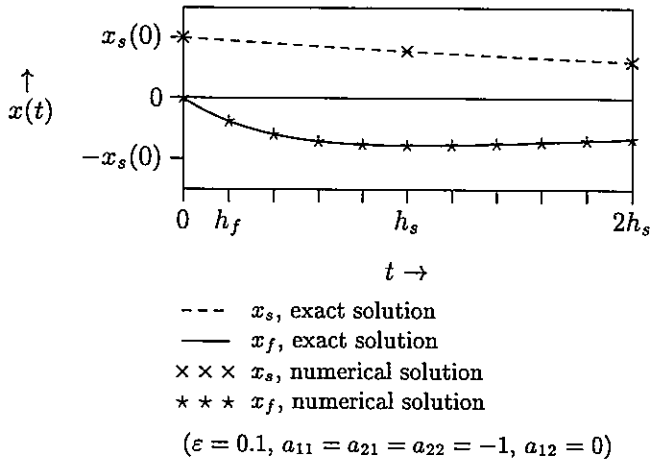


Fig. 2. Multiple step size algorithm.

steady state behavior of the original continuous time system has to be represented correctly by the derived discrete time system.

IV. Large Time Scale Analysis of Fast Subsystem

In parts of the network which are free of harmonics in steady state, computation time can drastically be reduced if an integration method is used which permits to increase the integration step size h_f, ideally up to $h_f = h_s$, when fast transients have decayed.

Those integration methods must satisfy the following conditions:

- Correct representation of fast transients for small integration step sizes.
- Frequency response at all occurring steady state frequencies is exact, even for large integration step sizes.
- A-stability, because integration step size will be much higher than time-constants of the fast subsystem.

Only methods up to second order can be A-stable. Hence, it is sufficient to consider these methods in this paper. Further, single step methods are focused because they can much easier be incorporated in an adaptive step size algorithm as required here.

A general single step integration method can be described by:

$$x(t+h) = x(t) + c_1\dot{x}(t) + c_2\dot{x}(t+h) \qquad (10)$$

The factors c_1 and c_2 depend on h.

Equation (10) corresponds to the following mapping between complex frequencies of the continuous time system p_c and of the discrete time system p:

$$p_c(p) = \frac{e^{ph} - 1}{c_2 e^{ph} + c_1} \qquad (11)$$

Using $z = e^{ph}$ the following expression can be obtained out of (11):

$$p_c(z) = \frac{z - 1}{c_2 z + c_1} \qquad (12)$$

Trapezoidal rule is obtained with $c_1 = c_2 = \frac{h}{2}$. Using $c_1 = c_2$ in (11) [2], the imaginary axis $p = j\omega$ of the p plane is entirely mapped to the imaginary axis $p_c = j\omega_c$ of the p_c plane.

Fig. 3 shows that the trapezoidal rule is exact around $\omega = 0$ and that its error grows for rising frequencies. The error depends further on integration step size h: The larger h, the faster the error grows for rising frequencies. Bandwidth of analyzed signals can therefore be much larger in case of a small integration step size h.

Trapezoidal rule is well suited for analyzing transients of the fast subsystem with small step sizes h_f, but it only complies with large time scale approximations for large h_f if the electrical network is balanced and if it is described in a rotating reference frame.

A. abc-Phase Domain

In absence of transients, state variables of the fast subsystem are of the type:

$$x_f(t) = a(t)\cos(\omega_n t) + b(t)\sin(\omega_n t)$$

The course of $a(t)$ and $b(t)$ depends on transients in the slow subsystem. The spectrum $\underline{X}_f(\omega)$ of $x_f(t)$ has bandpass characteristic with network frequency ω_n as central frequency and with a narrow bandwidth related to slow subsystem eigenvalues.

Therefore, trapezoidal rule must be modified in such a manner that the error is minimal around $\omega = \omega_n$. It can easily be verified that this can be obtained using:

$$c_1 = c_2 = \frac{\tan\left(\omega_n \frac{h}{2}\right)}{\omega_n} \qquad (13)$$

Fig. 3 shows that precision of this modified trapezoidal rule does not depend any more on absolute signal frequency but only on signal bandwidth around $\omega = \omega_n$.

Another convenient property of this modified trapezoidal rule can be shown using $\tan x \approx x$ for $|x| \ll 1$:

$$c_1 = c_2 \approx \frac{h}{2} \quad \left(\frac{\omega_n h}{2} \ll 1\right) \qquad (14)$$

Hence, the modified trapezoidal rule can, together with small h, also be used for analyzing transients of the fast subsystem.

B. Rotating Reference Frame

According to large time scale approximations, complex space phasors of fast state variables $\underline{x}_f(t)$ described in a

[2] this is even valid for $\underline{c}_1 = \underline{c}_2^*$ in case of complex integration methods

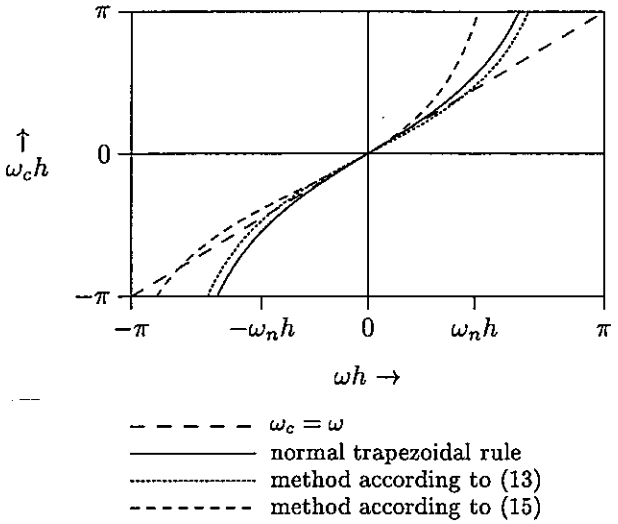


Fig. 3. Integration methods in frequency domain

rotating reference frame are of the following type [4], [5]:

$$\underline{x}(t) = \underline{a}(t) + \underline{b}(t)\mathrm{e}^{-j2\omega_n}$$

As before, $\underline{a}(t)$ and $\underline{b}(t)$ are related to transients of the slow subsystem [3].

In this case, an integration method must be found which is exact around $\omega = 0$ and around $\omega = -2\omega_n$. The following equations for $\underline{c}_1$ and $\underline{c}_2$ accomplish with this condition (see Fig. 3):

$$\underline{c}_1 = \underline{c}_2^* = \frac{h}{2}\left(1 + j\left(\cot(\omega_n h) - \frac{1}{h\omega_n}\right)\right) \qquad (15)$$

Regarding (11) and using $\underline{c}_1$ and $\underline{c}_2$ according to (15), it can easily be verified that this integration method has fixpoints at $p = p_c = 0$ and $p = p_c = -2j\omega_n$, independently of h.

Like the modified trapezoidal rule (14), this complex method approximates trapezoidal rule for small integration step sizes h and is therefore suited for analyzing fast transients, too.

Zero sequence components are of the same type as state variables in abc-phase coordinates. Consequently, they can be treated by (13).

The two described methods are appropriate for analyzing the complete system behavior including fast transients. When fast transients are decaying, integration step size can be increased without changing the integration method or the network model. The transition from small to large time scale can therefore be made smoothly without any parasitic excitation. Because $\underline{c}_1$ and $\underline{c}_2$ are constant for constant h, the described integration methods can easily be incorporated in a simulation algorithm of the Dommel type [3].

[3] in balanced networks: $\underline{b}(t) = 0$

It is important to mention that both methods can be used for balanced and unbalanced network states.

C. A-Stability

Only A-stable methods are appropriate for large time scale analysis of the fast subsystem. A-stability can easily be verified by describing the integration methods as a mapping of complex frequencies like (12).

An integration method is then A-stable if, for any value of h, the left p-half-plane is mapped into the unit-circle of the z-plane.

According to (12) each pole p_0 of the continuous time system is mapped to a pole z_0 of the corresponding discrete time system:

$$z_0 = \frac{1 + p_0\underline{c}_1}{1 - p_0\underline{c}_2} \qquad (16)$$

In order to be A-stable, a single step integration method must accomplish with:

$$\left|\frac{1 + p_0\underline{c}_1}{1 - p_0\underline{c}_2}\right| < 1 \qquad \text{for } \mathrm{Re}(p_0) < 0 \qquad (17)$$

Using $p_0 = \sigma_0 + j\omega_0$ and considering $\underline{c}_2 = \underline{c}_1^*$ with $\underline{c}_1 = c_r + jc_i$, condition (17) corresponds to:

$$(1 - \sigma_0 c_r - \omega_0 c_i)^2 + (\omega_0 c_r - \sigma_0 c_i)^2 - (1 + \sigma_0 c_r - \omega_0 c_i)^2 - (\omega_0 c_r + \sigma_0 c_i)^2 > 0 \quad \text{for } \sigma_0 < 0$$

This condition can further be simplified to:

$$-4\sigma_0 c_r > 0 \qquad \text{for } \sigma_0 < 0 \qquad (18)$$

Condition (18) holds for $c_r > 0$. The modified trapezoidal rule according to (13) is therefore stable if h lies in the following ranges (with $c_r = c_1 = c_2$, $c_i = 0$):

$$\omega_n \frac{h}{2} \in \left(k\pi, k\pi + \frac{\pi}{2}\right) \equiv h \in \left(k\frac{1}{f_n}, \left(k + \frac{1}{2}\right)\frac{1}{f_n}\right) \qquad \text{(k integer)} \quad (19)$$

Because these stability ranges are independent of system eigenvalues, the modified trapezoidal rule can be used for integrating state variables of the fast subsystem with large integration step sizes.

For $k = 0$, equation (19) corresponds to Nyquists sampling theorem for f_n. But stability ranges can also be found for $k > 0$. Theses cases, in which state variables are 'subsampled' will be analyzed in section V.

With (18), the complex integration method according to (15) is A-stable because $c_r = h/2 > 0$.

In unbalanced network states, there exists usually a zero sequence component which has to be treated according to (13). Hence, even if the network is described in a

rotating reference frame, h has to be chosen according to (19).

V. Control of Integration Step Size

Normally, integration step size is controlled by considering limits for the local truncation error.

No easy formula for estimating the local truncation error could be found for the described integration methods. Therefore, predictor-corrector schemes have been used for error estimation: Using an explicit predictor formula which makes the same considerations concerning signal waveforms like the corresponding implicit integration method, the local truncation error can be estimated and step size can be adjusted accordingly.

In addition, some natural limits for h are given by (19).

A. Nyquists Sampling Theorem for Bandpass Signals

Nyquists sampling theorem for general signals would require the following relation between the integration step size h_f and the maximum signal frequency f_{max}:

$$h_f > \frac{1}{2f_{max}} \tag{20}$$

But knowing that signals have bandpass character with bandwidth f_B and central frequency f_n, the sampling theorem can be reformulated:

$$h_f > \frac{1}{f_B} \tag{21}$$

In large time scale analysis, f_B of fast subsystem state variables only depends on eigenvalues of the slow subsystem. Therefore, and because (21) is independent of f_n, the maximum possible value of h_f also depends only on slow subsystem eigenvalues.

B. Influence on Global System Stability

Upper limits for h_f can only be given by regarding the global system behavior, including mechanical equations:

Considering that the central frequency of fast subsystem state variables is directly related to generator speed which is also state variable, the global system behavior must be described by a set of nonlinear differential equations which also takes mechanical equations into account:

$$\dot{\vec{x}} = \vec{g}(\vec{x}) \tag{22}$$

Equation (22) is built according to large time scale approximations, therefore, fast subsystem equations are only represented by their steady state behavior.

By linearizing (22), local stability of the global system can be analyzed. Supposing that the state variable x_n corresponds to the speed of a generator to which central frequency of fast subsystem state variables is directly related ($\omega_n x_n = \omega_c$), the according components of the linear system matrix $\mathbf{A_c}$ are:

$$a_{in_c} = \frac{\partial g_i}{\partial x_n} \quad (x_n = 1)$$

Considering the steady state behavior of the fast subsystem discretized by one of the described integration methods, together with the original, continuous time slow subsystem, the elements a_{in_d} of the according system matrix $\mathbf{A_d}$ can be obtained as follows:

$$a_{in_d} = a_{in_c} \frac{\partial \omega_c}{\partial \omega} \quad (x_n = 1)$$

Therefore, the derivative of the numerical integrator at $\omega = \omega_n$, in case of an abc-phase description, or at $\omega = 0$, to which $x_n = 1$ is related in case of a rotating reference frame, is of great interest. Using the mapping according to (11) the derivative can be calculated:

$$\frac{\partial p_c}{\partial p} = h e^{ph} \frac{\underline{c}_1 + \underline{c}_2}{\left(\underline{c}_2 e^{ph} + \underline{c}_1\right)^2} \tag{23}$$

Applied to the modified trapezoidal rule (13) this expression has to be analyzed at $p = j\omega_n$:

$$\left.\frac{\partial p_c}{\partial p}\right|_{p=j\omega_n} = \frac{\omega_n h}{\sin(\omega_n h)} \tag{24}$$

Therefore, only for $\omega_n h \ll 1$, global system stability is not influenced. At least, h should not exceed $h = \frac{1}{2f_n}$.

Evaluating (23) for the complex method according to (15) it can easily be shown that

$$\left.\frac{\partial p_c}{\partial p}\right|_{p=0} = 1$$

holds.

Here, the low frequency component which is related to positive sequence is not subsampled if h_f is chosen according to (21). Only components related to negative and zero sequence components are subsampled but their influence on global system stability can be neglected.

VI. Example

A typical two-machine problem has been chosen in order to test the described methods (see Fig. 5). Synchronous machines have been modeled according to Park's equations including voltage regulators.

The line model is a simple Π-element.

In order to demonstrate the performance of the presented methods, the network is unbalanced, even under normal operation conditions.

At $t = 0$, a single phase to ground fault is introduced in phase **a** at bus **b3** which is extended to a double phase to ground fault in phases **a** and **b** 30ms later.

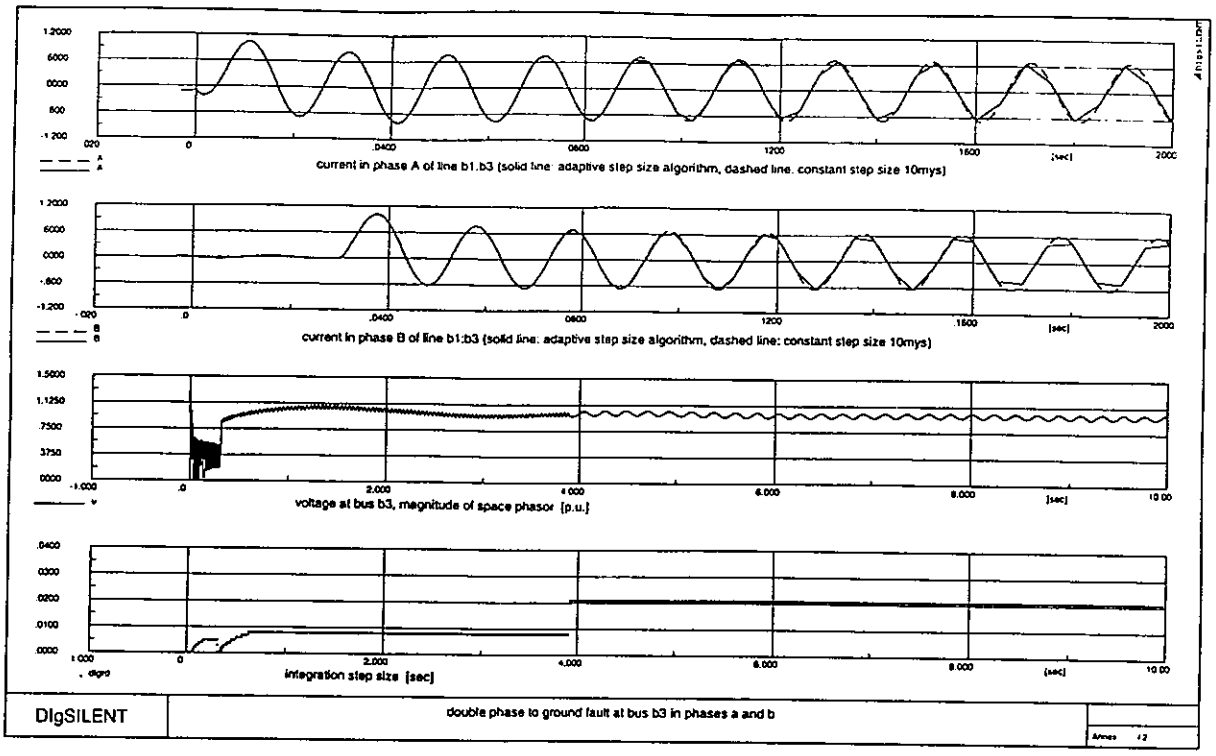


Fig. 4. Simulation results

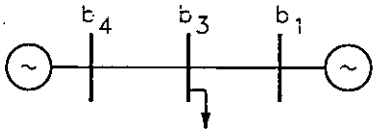


Fig. 5. Single line graphic diagram of test network

The duration of the short circuit is about $300ms$.

Fig. 4 shows simulation plots as they are produced by the simulation algorithm implemented in DIgSILENT without any post-treatment except for linear interpolation.

When integration step size is enlarged, phase currents don't seem to be sinusoidal any more. But this is only due to linear interpolation because all samples have still correct values as a comparison to the corresponding result of a standard EMT-simulation with $10\mu s$ integration step size proves (see Fig. 4).

The frequency caused by negative sequence components seems to change when integration step size *dtgrd* exceeds $20\text{ms} = 1/f_n$. But its amplitude and phase is still correct which shows that subsampling can be applied here.

VII. Conclusions

Linear, single step integration methods have been presented which comply with large time scale approximations for large integration step sizes h even in case of unbalanced electrical networks.

For small integration step sizes h, the described methods approximate trapezoidal rule and are therefore appropriate for calculating fast transients, too.

Using a rotating reference frame, only time constants of the slow subsystem limit integration step size after fast transients have decayed.

If the network is described in abc-phase coordinates, subsampling can lead to global system instability and can therefore only be used in case of one-machine problems.

In regions where nonlinear network elements produce considerable harmonics, the only way to profit by the multiple time scale character of electrical power systems is the use of standard multiple step size algorithms.

References

[1] J. H. Chow. *Time Scale Modeling of Dynamic Networks with Applications to Power Systems.* Springer-Verlag, Berlin/Heidelberg/New York, 1982.

[2] M. L. Crow and J. G. Chen. The multirate simulation of facts devices in power system dynamics. *IEEE Transaction on Power Systems*, 11(1), February 1996.

[3] H. W. Dommel. EMTP theory book. Technical report, University of British Colombia, 1986.

[4] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff. *Analysis of Electric Machinery.* IEEE Press, originally Mc Graw-Hill, 1995.

[5] B Oswald. *Netzberechnung 2, Berechnung transienter Vorgänge in Elektroenergieversorgungsnetzen.* VDE Verlag, Berlin - Offenbach, 1996.