On the Image Approximation of the Sommerfeld Integrals for Study of Fast Transients on Overhead Lines

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Abstract - **The characteristics of wave propagation guided by a single wire suspended in air above a lossy ground is studied with special regard to closed-form approximation of the correction terms for the electric scalar potential and all components of the magnetic vector potential. The propagation constant is addressed as is the difference between scalar potential and voltage for higher frequencies. Approximations for the complete electric and magnetic fields are discussed briefly.**

Keywords: Transmission Line, Wave Propagation, Fast Transients, Lossy Ground, Propagation Constant, Characteristic Impedance, Image Approximation, Quasi-TEM Approximation.

I. INTRODUCTION

In the study of fast transients on overhead transmission lines, one definite difficulty is realistic modelling of the influence of the ground. The effect of the finite conductivity of the ground is to cause resistive losses which are manifested in attenuating transients travelling along the line. Though much effort has been spent on the problem for almost a century, knowledge is still incomplete even for the generic case with a single straight wire suspended at a fixed height in air above a ground represented by a half space of homogenous electric properties. In analyses of low-frequency transients it may do to include a correction term, relative to the ideal case with an infinitely conductive ground, for the longitudinal effects only, as shown by Carson and Pollaczek [1,2]. For more high-frequency transients correcting has to be done also on the transverse effects as was shown by Wise [3]. For assessment of the voltage of the wire, a third correction for the vertical electric field between ground and wire will moreover be necessary. As a matter of fact an ensamble of correction terms will come in if the complete E- and B-field is studied. All these correction terms will be of integral type and are in a general sense called Sommerfeld integrals after the solution for the ground effects on an oscillating vertical dipole [4].

The corrections thus proposed are in fact only approximations as they have resulted from replacing an unknown propagation constant occurring in the integrals by that of free space. Computer codes for transients analyses would in spite of this approximation still contain time-consuming integrations, and simple closed-form reasonably accurate approximations are wanted. One such is the image approximation of Sunde and Kostenko [5,6] for the longitudinal effects, later generalized in [7,8] for the transverse effects and in this paper further exploited for approximating the vertical E-field and the voltage. It should be pointed out that the approximation of concern goes by many names such as the complex-depth or -plane method or the logarithmic approximation, see [7] for a historic review. Regarding the exact theory, Kikuchi [9] seems to have done the pioneering work. Frequently referenced papers on the exact theory are [10] and [11].

II. BASIC THEORY

A. Problem formulation

We study the propagation of waves guided by an open wave guide consisting of a single wire of radius a suspended at height h in air above ground which is seen as an open half space of finite conductivity σ, relative dielectric constant ε_r and permeability μ_0, see Figure 1. The wire is assumed to be thin, i.e. $h >> a$. The structure is assumed to have been excited by some distant harmonic source and we consider the propagation to be free of external sources. The fields are assumed to vary as $exp(-\gamma z)$ and we write the current in the

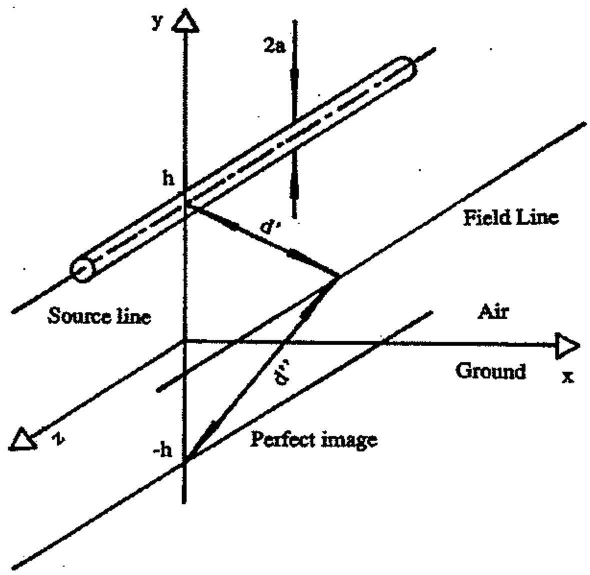

Figure 1 Wire Above Ground

line as $I=I_0 exp(-\gamma z+j\omega t)$, where t is time and ω angular frequency. The propagation constant γ is à priori unknown and has to be determined.

B. The potentials

It will be most facilitating to work with potentials and here we will make use of the electric scalar potential V and the magnetic vector potential $A=(A_x, A_y, A_z)$. Supposing γ to be given, the potentials in the air are

$$V(x,y)=I[\gamma/(j\omega\varepsilon_0)][\Lambda(x,y)+Q(x,y)]/(2\pi) \quad (1a)$$

$$A_x(x,y)=0 \quad (1b)$$

$$A_y(x,y)=I\frac{1}{j\omega}[\gamma/(j\omega\varepsilon_0)]R(x,y)/(2\pi) \quad (1c)$$

$$A_z(x,y)=I\mu_0[\Lambda(x,y)+P(x,y)]/(2\pi) \quad (1d)$$

with $\Lambda(x,y)=K_0(\gamma_\tau d'(x,y))-K_0(\gamma_\tau d''(x,y))$

$$P(x,y)=\int_{-\infty}^{\infty}\frac{E(x,y)}{u_1+u_2}dk \quad Q(x,y)=\int_{-\infty}^{\infty}\frac{E(x,y)}{n^2u_1+u_2}dk$$

$$R(x,y)=-\int_{-\infty}^{\infty}\frac{(u_2-u_1)E(x,y)}{n^2u_1+u_2}dk$$

where $E(x,y)=exp[-u_1(h+y)-jkx]$

$$\gamma_\tau=\left(\gamma_0^2-\gamma^2\right)^{1/2} \quad \gamma_0=j\omega(\varepsilon_0\mu_0)^{1/2}$$

$$u_1=\left[k^2+\gamma_0^2-\gamma^2\right]^{1/2} \quad u_2=\left[k^2+(n\gamma_0)^2-\gamma^2\right]^{1/2}$$

$$n=[\varepsilon_r+\sigma/(j\omega\varepsilon_0)]^{1/2}$$

$$d'(x,y)=\left[(h-y)^2+x^2\right]^{1/2} \quad d''(x,y)=\left[(h+y)^2+x^2\right]^{1/2}$$

Correspondingly, the potentials in ground may be given. These potentials satisfy the Helmholz equation in conjunction with the Lorenz gauge condition, vanish at infinity, have the right singularities at the line source, and give E- and B-fields that satisfy the boundary conditions on the ground/air surface. The quantity γ_0 is the intrinsic propagation constant of air, and γ_τ the transverse propagation constant. Further, n is the refractive index of ground so that $n\gamma_0$ is the intrinsic propagatin constant of ground. K_0 is the modified Bessel function of the second kind of order zero. All square roots are defined so that the real part is positive. This solution is due to Kikuchi [9] with A_y reconstructed.

In (1), the term Λ captures the source and the perfect image fields in the idealistic situation with a perfectly conducting ground, in which case P, Q and R will be zero. The latter three terms will be different from zero in the general case and represent correction terms due to a non-perfectly conducting ground.

C. The immitances

We consider now a point on the surface of the wire, to be specific we choose $x=0$, $y=h-a$. We can introduce the unit length series impedance Z and shunt admittance Y of the wire so that the "telegrapher's equations"

$$ZI=\gamma V \quad YV=\gamma I \quad (2)$$

are satisfied. For the matter of simplicity in denotations, the argument $(0,h-a)$ is understood. They are in fact given by

$$Z=j\omega\mu_0(\Lambda+P)/(2\pi) \quad Y^{-1}=\frac{1}{j\omega\varepsilon_0}(\Lambda+Q)/(2\pi) \quad (3)$$

where the condition that E_z be zero on the surface of the wire has been invoked since we assume that the wire is lossless. For a wire with losses a proper term representing the internal series impedance of the wire should be added to Z.

D. The modal equation

For (2) to have a solution it is required that

$$\gamma=(ZY)^{1/2}=\gamma_0[(\Lambda+P)/(\Lambda+Q)]^{1/2} \quad (4)$$

where it is understood that the right hand side is for the given point on the line, and moreover is a function of γ. This so called modal equation determines γ.

E. The characteristic impedance

The characteristic impedance Z^c of the wire defined by $Z^c=V/I$ is from (2) and (3)

$$Z^c=(Z/Y)^{1/2}=Z_0[(\Lambda+P)(\Lambda+Q)]^{1/2}/(2\pi) \quad (5)$$

$$Z_0=(\mu_0/\varepsilon_0)^{1/2}$$

which is completely determined when γ is determined. Here, Z_0 is the intrinsic impedance of air.

F. The fields

Now, the fields can be determined at any given field point in the air by applying $E=-\nabla V-j\omega A$ and $B=\nabla\times A$ on (1). It turns out that in total nine Sommerfeld integrals are involved for a complete field characterization. These can be written as

$$S(x,y)=\int_{-\infty}^{\infty}S'E(x,y)dk \quad (6)$$

with $E(x,y)$ as before and with

$$S_1' = 1/(u_1+u_2) \quad S_{11}' = k/(u_1+u_2) \quad S_{12}' = u_1/(u_1+u_2)$$
$$S_2' = 1/\left(n^2u_1+u_2\right) \quad S_{21}' = k/\left(n^2u_1+u_2\right)$$
$$S_{22}' = u_1/\left(n^2u_1+u_2\right) \quad S_{23}' = u_2/\left(n^2u_1+u_2\right)$$
$$S_{212}' = ku_1/\left(n^2u_1+u_2\right) \quad S_{213}' = ku_2/\left(n^2u_1+u_2\right)$$

The denotation system is designed so that the first index denotes the type of the denominator and that 1 in the second position comes after derivation with respect to x and 2 with respect to y. The same applies to the third index plus that 3 means u_2.

G. Voltage

The voltage U of the wire is by definition

$$U = -\int_0^{h-a} E_y dy = V - V_0 + j\omega \int_0^{h-a} A_y dy \qquad (7)$$
$$= I\frac{\gamma}{j\omega\varepsilon_0}\left[\Lambda + Q - Q_0 + T\right]/(2\pi) \qquad T = \int_0^{h-a} R(0,y)dy$$

where $V_0=V(0,0)$ and $Q_0=Q(0,0)$. Then we are in a position to write the characteristic impedance $Z_U^c = U/I$ for the voltage of the wire as

$$Z_U^c = Z_0\left(\frac{\Lambda+P}{\Lambda+Q}\right)^{1/2}\left(\Lambda+Q-Q_0+T\right)/(2\pi) \qquad (8)$$

$$T = \int_{-\infty}^{\infty}\frac{u_2-u_1}{u_1\left(n^2u_1+u_2\right)}\left(e^{-2hu_1}-e^{-hu_1}\right)dk$$

Without substantial loss of precision, a has been left out in comparison with h in T, and the same is done for P and Q.

III. APPROXIMATIONS

A. The quasi-TEM approximation

TEM means Transverse Electric and Magnetic fields which occur only if σ is infinite, in which case P and Q both become zero and γ_0 becomes the solution of the modal equation. The quasi-TEM approximation means that γ is set equal to γ_0 in P and Q after which the modal equation (4) is directly solved for the left hand side. We denote this approximation by superscript "~". Specifically we have

$$\tilde{\Lambda} = \ell n(2h/a) \quad \tilde{P} = \int_{-\infty}^{\infty}\frac{e^{-2h|k|}}{|k|+\left(k^2+\beta^2\right)^{1/2}}dk \qquad (9)$$
$$\tilde{Q} = \int_{-\infty}^{\infty}\frac{e^{-2h|k|}}{n^2|k|+\left(k^2+\beta^2\right)^{1/2}}dk$$
$$\tilde{T} = \int_{-\infty}^{\infty}\frac{\left[\left(k^2+\beta^2\right)^{1/2}-|k|\right]\left(e^{-2h|k|}-e^{-h|k|}\right)}{|k|\left[n^2|k|+\left(k^2+\beta^2\right)^{1/2}\right]}dk$$

with $\beta = \gamma_0\left(n^2-1\right)^{1/2}$

which are to be used in (4) and (5). Historically, $\tilde{P}$ was found independently by Pollaczek [2] and Carson [1] and $\tilde{Q}$ by Wise [3]. The origin of $\tilde{T}$ is unknown to this author.

For the complete field characterization, (6) stands almost as it is just by replacing u_1 by $|k|$ and u_2 by $\left(k^2+\beta^2\right)^{1/2}$.

B. The image approximation

By image approximation is here meant that the quasi-TEM approximation is further approximated according to a certain procedure to yield closed-form formulae. The basic idea of the procedure is to approximate the S'-factors of (6) in a way that makes exact integration possible. The main vehicle is the approximation

$$\left[bk+\left(k^2+\beta^2\right)^{1/2}\right]^{-1} \approx \frac{1}{k}\left[1-exp\left(-k\frac{1+b}{\beta}\right)\right]/(1+b) \qquad (10)$$

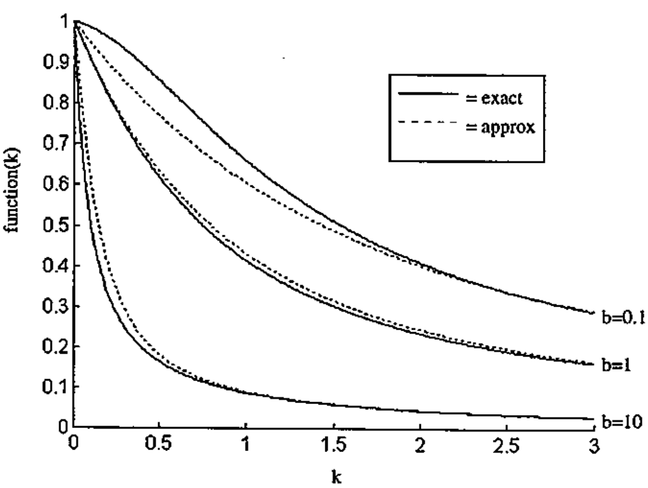

Figure 2 Precisison of the basic approximation

It is easily verified that the two sides agree asymptotically for small as well as large values for k/β, which so far makes the approximation promising. In Figure 2, the precision of the approximation is further probed for the case with both b and β real and positive. Here $\beta = 1$ is assumed without loss of generality, and $b=0.1$, 1 and 10 are used for illustration. The maximum relative error for $b=1$ is +3%. For $b<1$, the figure is -12%, which is attained for $b=0$, and for $b>1$ +30% for b approaching infinity.

For application on S_1 and S_2, i.e. P and Q, we set $b=1$ and $b=n^2$, respectively. Noting that with $c=(1+b)/\beta$ we have

$$\left(1-e^{-kc}\right)/k = \int_0^c e^{-kw} dw$$

which used in the integrands of (9), after reversing the order of integration gives as a result

$$\int_0^c \int_0^\infty exp\left[-k(w+h+y \pm jx)\right] dw dk = \int_0^c (w+h+y \pm jx)^{-1} dw = \ell n\left(\frac{c+h+y \pm jx}{h+y \pm jx}\right)$$

giving, with superscript "≈" denoting the image approximation

$$\tilde{\tilde{S}}_{1,2} = \frac{2}{1+b} \ell n \left\{ \left[\left(h+y+\frac{1+b}{\beta}\right)^2 + x^2 \right] / \left[(h+y)^2 + x^2 \right] \right\}^{1/2}$$

We can thus write

$$\tilde{\tilde{P}}(x,y) = \ell n(d_P / d'') \quad d_P = \left[\left(h+y+\frac{2}{\beta}\right)^2 + x^2 \right]^{1/2} \tag{11}$$

$$\tilde{\tilde{Q}}(x,y) = \frac{2}{n^2+1} \ell n(d_Q / d'')$$

$$d_Q = \left[\left(h+y+\frac{n^2+1}{\beta}\right)^2 + x^2 \right]^{1/2}$$

Here $\tilde{\tilde{P}}$ *(x,y)* is due to Kostenko [6] and the generalization of the method giving $\tilde{\tilde{Q}}$ *(x,y)* to this author [7,8].

For use in the characteristic impedance, (11) reduces into

$$\tilde{\tilde{P}} = \ell n\left(1+\frac{1}{\beta h}\right) \quad \tilde{\tilde{Q}} = \frac{2}{n^2+1} \ell n\left(1+\frac{n^2+1}{2\beta h}\right) \tag{12}$$

$$\tilde{\tilde{Q}}_0 = \frac{2}{n^2+1} \ell n\left(1+\frac{n^2+1}{\beta h}\right)$$

where $\tilde{\tilde{P}}$ was first suggested by Sunde [5] without a formal proof.

Proceeding to approximate R and T of (7) we have to approximate S_{23}-S_{22} in (6). Now using again (10) but now with $b=0$ we have that

$$u_2 = \left(k^2+\beta^2\right)^{1/2} \approx k/\left(1-e^{-k/\beta}\right)$$

Further using the following approximation which is asymptotically correct for large k and which is correct up to two terms in a series expansion for small k

$$\left(1-e^{-kb}\right)/\left(1-e^{-kc}\right) \approx \frac{b-c}{c} exp\left(-k\frac{b-c}{2}\right)+1$$

with $b=\left(n^2+1\right)/\beta$ and $c=1/\beta$ gives, using again (10) for the u_1-term

$$\tilde{\tilde{R}}(x,y) = \frac{n^2}{n^2+1} \frac{2\left(y+h+\frac{n^2}{2\beta}\right)}{\left(y+h+\frac{n^2}{2\beta}\right)^2 + x^2} + \frac{1}{n^2+1} \frac{2\left(y+h+\frac{n^2+1}{\beta}\right)}{\left(y+h+\frac{n^2+1}{\beta}\right)^2 + x^2} \tag{13}$$

By (11) and (13), all potentials in (1) have been given closed-form approximations.

Setting $x=0$ in $\tilde{\tilde{R}}(x,y)$ and performing the integration from $y=0$ to h yields

$$\tilde{\tilde{T}} = -\frac{2}{n^2+1}\left(n^2 \ell n \frac{2+\frac{n^2}{2\beta h}}{1+\frac{n^2}{2\beta h}} + \ell n \frac{2+\frac{n^2+1}{\beta h}}{1+\frac{n^2+1}{\beta h}} \right) \tag{14}$$

Now, (12) and (14) together with $\tilde{\tilde{\Lambda}} = \ell n(2h/a)$ inserted into (8) determines the image approximation of the voltage of the wire. Actually some terms are cancelling so that

$$\tilde{\tilde{Q}} - \tilde{\tilde{Q}}_0 + \tilde{\tilde{T}} = -\frac{2}{n^2+1}\left(n^2 \ell n \frac{2+\frac{n^2}{2\beta h}}{1+\frac{n^2}{2\beta h}} + \ell n 2 \right) \qquad (15)$$

IV. VALIDATION

The validation of the approximations is confined to on one hand the propagation constant and on the other the characteristic impedances of the scalar potential and the voltage. Instead of working directly with these quantities, it is convenient to have them normalized to the corresponding quantities for an ideally perfectly conducting ground. We thus introduce α as the normalized propagation constant and ψ as the normalized characteristic impedance through

$$\gamma = \alpha\gamma_0 \qquad Z^c = \psi Z_0 \ell n(2h/a)/(2\pi)$$

Here Z^c and ψ apply to either scalar potential or voltage.

Both approximations are compared with the exact solution for the frequency range 0.05 - 50 MHz. The exact solution use iterations to solve the modal equation. Figure 2 shows the trajectories of the real and imaginary part of α, and Figure 3 and 4 the corresponding for ψ for scalar potential and voltage respectively. The following parameters are used: h=10 m, a=0.01 m, ε_r =10 and σ=1 mS/m.

Regarding α in Figure 3 it is seen that both approximations work rather good for the whole frequency range. Unexpectedly it appears that the image approximation can be better than quasi-TEM for some frequencies, especially in the higher frequency range. It should be pointed out that $-Im(\alpha)$ is a measure of the attenuation and $Re(\alpha)$ is the inverse of the phase velocity relative to light.

Regarding ψ, we notice for the scalar potential in Figure 4, that the approximations are rather close to each other for the whole range and they are close to the exact solution up to about 0.5 MHz. Above that, they differ markedly and quasi-TEM is slightly better than image.

For the voltage in Figure 5 the same can be said for the lower range. However, for the upper range the situation is more complex. For frequencies about 5 MHz the image approximation is better than quasi-TEM while the result at 50 MHz is inconclusive.

Comparing Figures 4 and 5 we note an agreement between the scalar potential and voltage formulation for the characteristic impedance at 0.05 MHz. For higher frequencies, however, there is an increasing difference. We see a markedly lower resistive and a higher capacitive component in the voltage than in the scalar potential, as manifested in $Re(\psi)$ and $Im(\psi)$, respectively. Most remarkable is the circumstance that the resistive component is always greater than unity for the scalar potential, while it can attain values below that for frequencies high enough. In effect, while the limiting point is the expected $1+j0$ for the scalar potential, it is $1-2\ell n\ 2/\ell n(2h/a) + j0 = 0.82 + j0$ for the voltage. To the authors knowledge, such a behaviour has not been reported in the literature.

The results of the recent paper [12] seem to differ from those of this paper. At the root of this may be a missing integral of the vertical magnetic vector potential wich leads to the scalar potential difference between wire and ground rather than the voltage. See further the Discussion in [12].

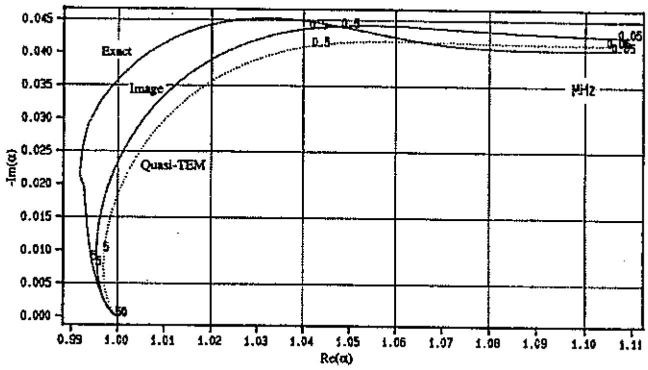

Figure 3 Relative propagation constant α of a wire above ground. h=10 m, a=0.01 m, ε_r=10, σ=1 mS/m

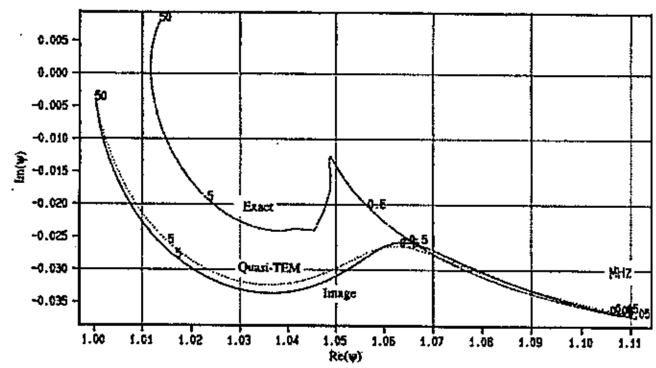

Figure 4 Relative characteristic impedance ψ of a wire above ground with a potential formulation. h=10 m, a=0.01 m, ε_r=10, σ=1 mS/m

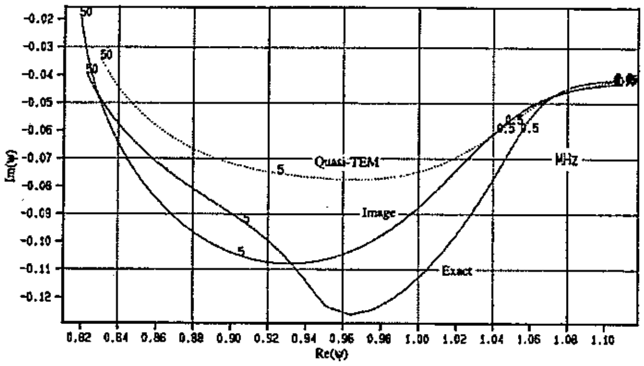

Figure 5 Relative characteristic impedance of a wire above ground with a voltage formulation.
h=10 m, a=0.01 m, ε_r=10, σ= 1 mS/m

V. CONCLUSIONS

With a voltage formulation for the wave propagation on a line above a lossy earth, an extra correction term is required in addition to the two pertaining to the scalar potential formulation earlier studied. This term, which comprises the vertical electric field effects, has been identified and has been found to be of Sommerfield integral type. The approximation scheme used earlier for the scalar potential formulation has been generalised in this paper to handle also this new integral. The precision of the extended approximation method has been tested for a typical overhead line geometry and has been found reasonably good.

In the cause of running the numeric comparisons, it surprisingly appeared that the limit for the characteristic impedance of the voltage for frequency tending to infinity was different from that of the scalar potential. For the latter, this impedance coincides as expected with the value when the ground is assumed perfectly conducting, while for the former a 18 % lower value was a attained.

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