Rational functions as frequency dependent equivalents for transient studies

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Abstract—**When analysing electromagnetic transients in localised parts of large power systems, it is common practice to represent the bulk of the AC system as a simple equivalent circuit. To obtain accurate transient results, it is necessary that the equivalent circuit mimic the frequency dependence of the system it is replacing. Present methods of designing such a frequency dependence are either not very accurate, require some skill to set-up, or employ difficult non-linear optimisation. This paper describes an automatic and robust procedure for finding single-port frequency dependent equivalents using *s*-domain rational functions. The rational function is found directly in the least squares sense using Singular Value Decomposition (SVD), and can be employed directly in transient studies.**

Keywords: Equivalents, Transient Analysis, *s*-domain Modelling, Rational Function.

I. Introduction

Power system transient studies require good models of transmission networks and components if accurate results are required. However it is prohibitive to represent large networks in their entirety for these studies due to the computational burden. This restricts complexity and hence reduces accuracy.

Using equivalents to model large networks and components for transient studies is now common. They increase solution speed, while reducing the memory requirement.

Traditionally, fundamental frequency Thevenin equivalents based on the short circuit impedance at a bus, have been used for AC system circuit breaker fault current design. For more detailed analysis of the effect of switchings and fault transients, equivalents which accurately represent the system impedance over a larger frequency range are required.

Hingorani and Burberry [1] proposed a technique to model the AC system frequency dependence for filter design and overvoltage studies at HVdc installations. The peaks and troughs of an AC system impedance were related to parallel and series resonances and realised as *RLC* networks.

The use of this method has remained largely unchanged, with various authors [2,3] applying least squares techniques to minimise error, in some cases successfully. More recently this technique was applied to represent multi-port equivalents for transmission networks [4], again using minimisation routines to eliminate approximation error.

Unfortunately this approach is both time consuming, due to the computationally intensive non-linear error minimisation required, and limited in its solution range, as the network is constrained to a particular configuration.

Rational function approximation was first proposed by Semlyen *et al* [5] to represent transmission line frequency dependence for transient studies, however the problem remained a non-linear one.

This paper presents a fast and stable alternative to determine single-port frequency dependent equivalents, using a rational function approximation in the *s*-domain. The problem is written as an overdetermined linear equation set, and solved directly in the least squares sense using Singular Value Decomposition (SVD).

The CIGRE Benchmark HVdc Rectifier AC test system [6] is utilised and equivalents derived using both the rational function and Hingorani network technique. Transient results are presented that compare the original and equivalent realisations using the electromagnetic transients package EMTDC [7].

Finally to demonstrate the robustness of the approach, a rational function approximation is applied to a more realistic system; the New Zealand lower South Island AC system from the Tiwai $220kV$ bus.

II. Hingorani equivalents method

The Hingorani based equivalents approach involves identifying parallel and series resonances from an impedance profile, and choosing suitable *RLC* component values within a set network configuration, to approximate the impedance.

The network configuration consists of parallelled second order *RLC* series branches, with an *RL* branch representing the response at $50Hz$. The *RLC* shunt branches are designed to series resonate at the impedance trough frequencies, with the shunt branch combinations parallel resonating at the impedance peaks. A generalised network configuration is shown in Fig.1.

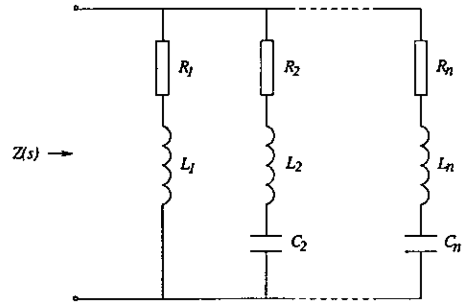

Fig. 1. General form of the Hingorani RLC equivalent network

This configuration has been further developed by some authors [2,5] to include a first order RC branch to model high frequency effects. While this method has remained the chosen approach for many, there are a number of problems associated with it. The first, concerns the time consuming nature of the non-linear minimisation routine required, due to $R_1 - R_n$, $L_1 - L_n$ and $C_2 - C_n$ affecting the approximation error function non-linearly.

The second problem involves the limited range of solutions that are possible for the constrained Hingorani network. This can be shown by decomposing, via partial fractions, a rational function of arbitrary numerator and denominator orders into the format of the n-branch RLC network (Fig.1) in admittance form, written as

$$Y(s) = \frac{p_n s^n + \ldots + p_2 s^2 + p_1 s + p_0}{s^m + \ldots + q_3 s^3 + q_2 s^2 + q_1 s + q_0}. \quad (1)$$

Given that (1) contains complex conjugate roots in the denominator (representative of series resonances), this may be written

$$Y(s) = \frac{U_1}{(s+\alpha_1)} + \frac{(V_1 s + W_1)}{(s^2 + \beta_1 s + \delta_1)} + \ldots + \frac{(V_r s + W_r)}{(s^2 + \beta_r s + \delta_r)}. \quad (2)$$

However $Y_h(s)$, the admittance of the n branch Hingorani network is written as

$$\begin{aligned} Y_h(s) &= \frac{\frac{1}{L_1}}{(s + \frac{R_1}{L_1})} + \frac{s\frac{1}{L_2}}{(s^2 + s\frac{R_2}{L_2} + \frac{1}{L_2 C_2})} + \ldots \\ &\quad \ldots + \frac{s\frac{1}{L_n}}{(s^2 + s\frac{R_n}{L_n} + \frac{1}{L_n C_n})}. \end{aligned} \quad (3)$$

The absence of terms W_1 to W_r in (3), illustrates that the RLC network in Fig.1 can only be a subset of the possible approximation solutions. By ignoring these terms, unnecessary error may be introduced.

In general, this approach models parallel and series resonant frequencies correctly, however there tends to be error in the damping at these resonances, leading to unreliable time domain results.

When equivalent networks form only part of the overall system, final system resonances may occur at any frequency, as a combination of all system components. In this instance a more stringent view of the approximation error is required.

III. Rational Function approximation

A. Approximation procedure

Approximating an impedance profile or frequency response using an s-domain rational function is possible in the frequency domain, due to the reversible nature of the Laplace and Fourier integral transforms in the steady state [8].

By substituting $j\omega$ for s, a rational function $\hat{G}(s)$, of arbitrary numerator and denominator orders m and n respectively, can be written in the frequency domain as

$$\hat{G}(j\omega) = \frac{b_0 + b_1(j\omega) + b_2(j\omega)^2 + \ldots + b_m(j\omega)^m}{1 + a_1(j\omega) + a_2(j\omega)^2 + \ldots + a_n(j\omega)^n}. \quad (4)$$

A natural approach to the approximation problem is to write an error function (5); the difference between $\hat{G}(j\omega)$ and the impedance $G(j\omega)$, and minimise it in the least squares sense over the frequency range k. The minimisation occurs in terms of the coefficients $b_0, b_1, \ldots, b_m$ and $a_1, a_2, \ldots, a_n$.

$$E = \sum_{k=0}^{p} |G(j\omega_k) - \frac{b_0 + b_1(j\omega_k) + \ldots + b_m(j\omega_k)^m}{1 + a_1(j\omega_k) + \ldots + a_n(j\omega_k)^n}|^2 \quad (5)$$

However this technique requires the use of a comparatively difficult non-linear optimisation method [9] which is time consuming.

Stahl [9] proposed the expression of the approximation as a set of linear equations which can be solved directly.

This is accomplished by decomposing $G(j\omega_k)$ into its real and imaginary components, c_k and d_k, for each frequency point $k = 0, 1, \ldots, p$, and equated with $\hat{G}(j\omega_k)$ as

$$(c_k + jd_k) = \frac{b_0 + b_1(j\omega_k) + \ldots + b_m(j\omega_k)^m}{1 + a_1(j\omega_k) + \ldots + a_n(j\omega_k)^n}. \quad (6)$$

This is further written as

$$(c_k + jd_k) = \frac{(b_0 - \omega_k^2 b_2 + \ldots) + j(\omega_k b_1 - \omega_k^3 b_3 + \ldots)}{(1 - \omega_k^2 a_2 + \ldots) + j(\omega_k a_1 - \omega_k^3 a_3 + \ldots)}. \quad (7)$$

The rational function coefficients become linear within the equation set when (7) is expanded to become

$$(c_k + jd_k)\left[(1 - \omega_k^2 a_2 + \dots) + j(\omega_k a_1 - \omega_k^3 a_3 + \dots)\right] = \left[(b_0 - \omega_k^2 b_2 + \dots) + j(\omega_k b_1 - \omega_k^3 b_3 + \dots)\right]. \quad (8)$$

The real and imaginary components are equated, resulting in 2 equations (9) and (10),

$$-d_k\omega_k a_1 - c_k\omega_k^2 a_2 + d_k\omega_k^3 a_3 + c_k\omega_k^4 a_4 - \dots \dots - b_0 + \omega_k^2 b_2 - \omega_k^4 b_4 + \dots = -c_k, \quad (9)$$

$$c_k\omega_k a_1 - d_k\omega_k^2 a_2 - c_k\omega_k^3 a_3 + d_k\omega_k^4 a_4 + \dots \dots - \omega_k b_1 + \omega_k^3 b_3 - \omega_k^5 b_5 + \dots = -d_k. \quad (10)$$

These equations are expressed in matrix form as the equation set $Ax = b$. The coefficients of the rational function are defined in x, with A and b constructed below.

$$A = \begin{pmatrix} -w_0 d_0 & -w_0^2 c_0 & \dots\lambda_{e0} & -1 & 0 & \dots\lambda_{g0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -w_k d_k & -w_k^2 c_k & \dots\lambda_{ek} & -1 & 0 & \dots\lambda_{gk} \\ w_0 c_0 & w_0^2 d_0 & \dots\lambda_{f0} & 0 & -w_0 & \dots\lambda_{ho} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_k c_k & w_k^2 d_k & \dots\lambda_{fk} & 0 & -w_k & \dots\lambda_{hk} \end{pmatrix} \quad (11)$$

where

$$\lambda_{ek} = (sin(\frac{-n\pi}{2})w_k^n d_k + cos(\frac{n\pi}{2})w_k^n c_k), \quad (12)$$

$$\lambda_{fk} = (cos(\frac{n\pi}{2})w_k^n d_k + sin(\frac{n\pi}{2})w_k^n c_k), \quad (13)$$

$$\lambda_{gk} = (cos(\frac{-m\pi}{2})w_k^m), \quad (14)$$

$$\lambda_{hk} = (sin(\frac{-m\pi}{2})w_k^m), \quad (15)$$

$$x = (a_1, \dots, a_n, b_0, \dots, b_m)^T, \quad (16)$$

$$b = (-c_0, \dots, -c_k, -d_0, \dots, -d_k)^T. \quad (17)$$

B. Condition and Solution

A high resolution approximation over a large or even moderate frequency range will result in $Ax = b$ being overdetermined, requiring non-standard matrix inversion. Singular Value Decomposition (SVD) was chosen due to the stability and least squares solution capability [10].

Matrix condition problems were encountered due to the non-orthogonal nature of the rational function, and were exacerbated by either a large approximation dynamic range or high function order.

Matrix condition, which is a measure of the magnitude ratio of the largest singular value to the smallest, indicates numerical confidence in a linear equation set solution. A badly conditioned problem will typically result in solution inaccuracy. Euclidean scaling was successfully employed as a re-conditioning technique and is outlined in Appendix A.

IV. Application of Rational Function Equivalents for Transient Studies

A. Approximation

For both the time domain validation of the s-domain rational function approach and a comparison with the existing Hingorani method, equivalents are found using the CIGRE Benchmark Rectifier AC system impedance [6] as a base case.

The CIGRE benchmark model, while originally designed for use in HVdc control studies, is useful for this transient validation, as it provides an easily implementable network of components for model verification.

Choosing a suitable rational function is dependent on approximation error, with the required level of accuracy determined by the user. This involves the function numerator and denominator orders increasing simultaneously and automatically until the specified error threshold is satisfied.

The Maximum Absolute Error (MAE) criteria, based on the maximum error at any frequency over the range of the approximation, was implemented. For the CIGRE system impedance approximation, the MAE with respect to increasing rational function numerator and denominator orders is shown in Fig.2.

For a 3 per cent MAE, the 5^{th} order approximation satisfied the criteria. At the higher orders, accuracy below 0.1 per cent MAE is possible.

The Hingorani RLC network approximation was found to contain an RL and two RLC branches, with component values detailed in Appendix B. These values were determined using ACREP, a program developed by Watson and Arrillaga [4] at the University of Canterbury, to find AC system equivalents for transient studies around HVdc converter installations.

A comparison of the two equivalents and the CIGRE impedance is shown in Fig.3. The rational function and CIGRE impedance are seen to be identical over this frequency range, while some difference is noted with the Hingorani approximation, particularly for the phase angle above $150Hz$, and the magnitude of the dominant parallel resonance at $100Hz$.

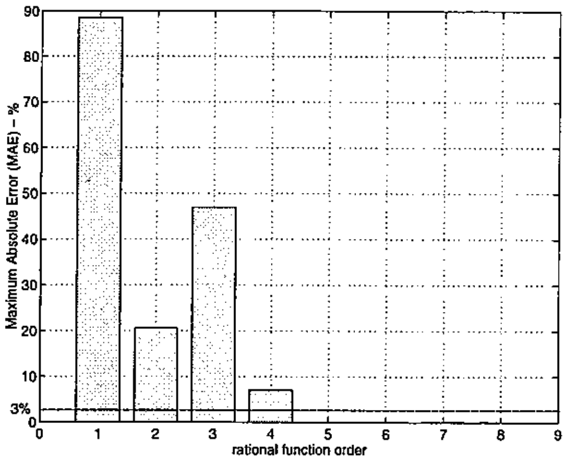

Fig. 2. Maximum Absolute Error for increasing rational function order ($m = n$)

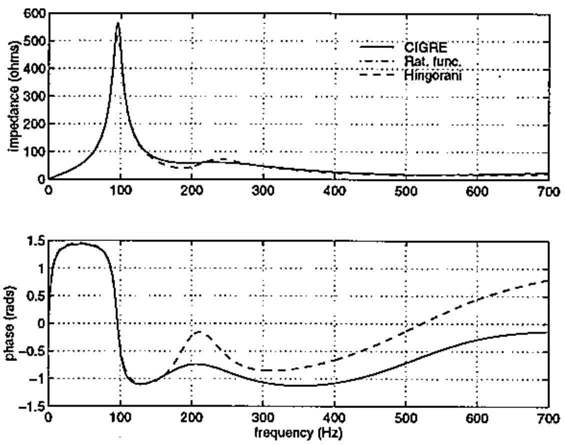

Fig. 3. Comparison of Hingorani and rational function equivalent techniques for the CIGRE Benchmark Rectifier AC system

B. Transient validation

The SVD solution yields an s-domain equivalent readily simulated in electromagnetic transient packages like EMTDC [7] and EMTP [11], which have s-domain defined functions available for control representation. They are implemented as differential equations, and solved by trapezoidal integration.

The transferral of variable information between the s and time domains during simulation incurs a single time step delay [11]. This delay will introduce phase error, particularly at high frequencies, and in the worst case may lead to numerical instability. A calculation time step should be chosen such that this error will be acceptable at the highest frequency of interest. Even so, in any system, numerical solution stability may not be guaranteed.

Model stability though can be determined prior to implementation through solution of the rational function denominator roots, however to date the method has yet to return an unstable approximation of an AC system impedance.

Simulations were carried out to compare voltage transients due to a current source input step perturbation for the rational function equivalent, Hingorani equivalent and base case CIGRE system impedance. The RLC networks had a current source directly connected with voltage transient information measured at the network input. The configuration for this is illustrated in Fig.4.

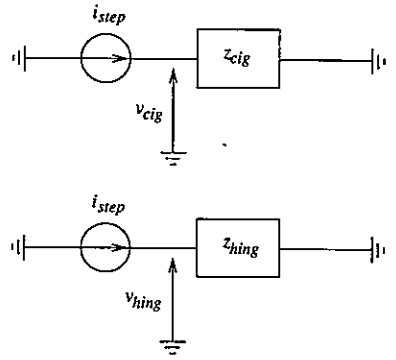

Fig. 4. Configurations for the CIGRE system and Hingorani equivalent for transient validation

For the rational function equivalent, current source information is transferred to the s-domain, defining the input $I(s)$ to the equivalent impedance $Z_{rat}(s)$. The system output $V_{rat}(s)$ is measured, and defined as a time domain variable.

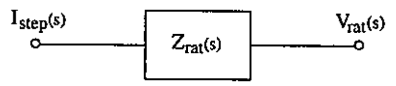

Fig. 5. Configuration of the rational function equivalent for transient validation

The transient results obtained using EMTDC are illustrated in Fig.6 for a DC current source step input from $0 - 1kA$ at time $t = 0.1secs$. The results indicate the expected $100Hz$ oscillation due to the lightly damped parallel resonance in the CIGRE system.

While the rational function and Hingorani equivalent results are similar, the rational function result more accurately reflects the CIGRE system voltage transient.

V. Application to New Zealand Lower South Island AC system

The application of the rational function equivalents technique to a realistic power system is detailed, with the New Zealand lower South Island test system [12] chosen. The impedance of the AC system as viewed from the Tiwai $220kV$ bus (see Fig.7) was generated using HARMAC

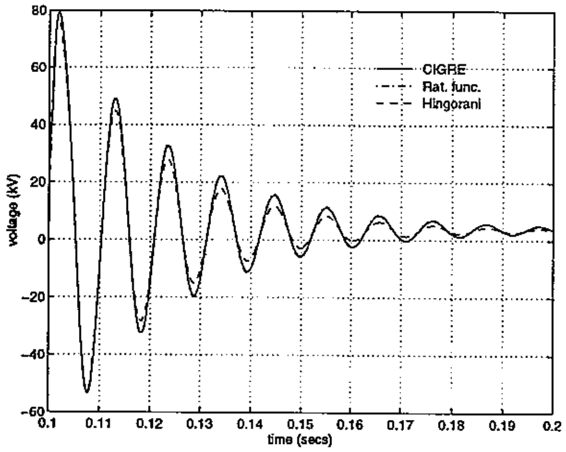

Fig. 6. Comparison of transient results for the Hingorani network, rational function and base case CIGRE system for a current source step input

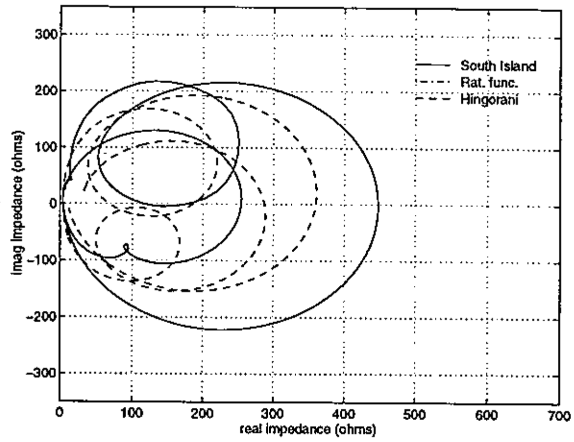

Fig. 8. Impedance loci of the South Island New Zealand AC system compared with the rational function and Hingorani approximations

[12], a three phase harmonic penetration program, which can produce impedance data at $5Hz$ increments.

The three-phase system impedance matrix is expressed in diagonal form which is suitable for single-port equivalents modelling only. The extension to a multi-port representation should present no serious difficulties as functions describing each cross-coupling term can be found in the s-domain, allowing each phase voltage to be determined as a function of the three phase currents.

The rational function ($m = n = 14$) and Hingorani equivalent (5 branch) approximations [12] of the phase B impedance up to $1250Hz$, are shown in loci form in Fig.8. Again the rational function approximation is virtually identical to the test system impedance.

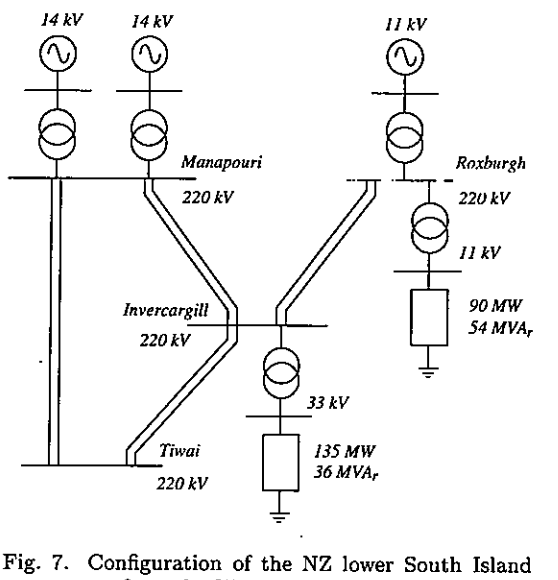

Fig. 7. Configuration of the NZ lower South Island test system from the Tiwai $220kV$ bus

VI. Conclusion

This paper has presented a fast and accurate technique for determining single-port frequency dependent equivalents for transient studies. The equivalents can be expressed as either voltage sources or voltage dependent current sources in the time domain.

The approximation of the system impedance is carried out in the frequency domain using an s-domain defined rational function, resulting in an overdetermined equation set. The equation set is solved directly in the least squares sense using Singular Value Decomposition. Matrix condition problems inherent in the approximation procedure are alleviated using Euclidean scaling on the overdetermined equation set.

Model order selection is based on a user defined error threshold and is automatic, with the resulting s-domain rational function readily implementable in electromagnetic transients packages.

The Hingorani *RLC* network equivalents approach has been discussed, outlining the non-linear nature of the approximation problem and the computationally intensive minimisation routine required. It has also been shown that the Hingorani network configuration is a subset of the possible solutions for any impedance.

The CIGRE Benchmark Rectifier AC test system is used as a basis for the Hingorani *RLC* network and rational function equivalents. Voltage transient results for a current source step input perturbation for the equivalents and base case CIGRE system are compared using EMTDC, with the rational function equivalent presenting more accurate results.

Finally, the application of the rational function method to the New Zealand lower South Island AC system phase B impedance as seen from the Tiwai $220kV$ bus, is presented. These results indicate the robustness and accuracy of the rational function approach when compared with the Hingorani approximation for this impedance.

VII. Acknowledgements

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VIII. References

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IX. Appendix

A. Euclidean scaling

Euclidean scaling is a method of reconditioning a set of linear equations to reduce the condition and hence improve solution accuracy. Given a matrix A_{pq}, where

$$A_{pq} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{pmatrix}. \tag{18}$$

The Euclidean length E_1 for the first column of A_{pq} is

$$E_1 = \| A_{p1} \| = \sqrt{a_{11}^2 + a_{21}^2 + \ldots + a_{p1}^2}. \tag{19}$$

The first column vector A_{p1} is rescaled by E_1, to create a new column vector $\hat{A}_{p1}$. A new matrix $\hat{A}_{pq}$ is then constructed from A_{pq}, with columns scaled by the Euclidean lengths. SVD is applied to the respecified problem, returning a parameter matrix $\hat{x}_q$. To return the true rational function coefficients, the terms in $\hat{x}_q$ are scaled by the corresponding Euclidean lengths from A_{pq},

$$(x_1, x_2, \ldots, x_q) = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_q)\, diag(1/E_q) \tag{20}$$

B. Hingorani RLC component values

The network components for the Hingorani approximation method of the CIGRE system impedance are,

$R_1 = 3.5\Omega$, $L_1 = 153.6mH$, $R_2 = 15.5\Omega$, $L_2 = 7.86mH$, $C_2 = 11.387\mu F$, $R_3 = 52\Omega$, $L_3 = 141.71mH$, $C_3 = 4.9\mu F$.