Modeling Power Systems with General Difference Equations - A Systematic Formulation

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Abstract - **Difference equations are widely used for analyzing power systems transients. Difference equations of power systems elements are systematically derived using different approaches and then combined to a difference equation system describing the complete electrical power system. By reordering these equations, a discrete time state space formulation is obtained. Hence, the complete theory of discrete time systems can be applied.**

Keywords: Simulation Methods, Difference Equations, Object Oriented Algorithms.

I. Introduction

Difference equations have been used for the analysis of power systems transients since the late sixties when Dommel introduced the so called *difference conductance method* [1].

The greatest advantage of difference equations methods is the simplicity of the solution process: By converting the continuous time equation of each branch element to a discrete time equation using e.g. the trapezoidal rule, each branch element can be described by either a current or voltage difference equation. The complete difference equation system can then be built without the necessity of identifying a set of independent state variables.

The difference conductance method is exclusively based on a current representation for all branch elements. Hence, for building the complete difference conductance system, the nodal voltage method, well known from network theory, can be applied.

Also just by applying Kirchoffs current law, different systems of difference equations can be built having not only node voltages as unknowns but also branch currents. Power systems elements can here be described either as current or voltage equations.

II. Continuous Time Equations of Power Systems Elements

The continuous time equations of power systems elements can either be described as differential or as integral equations. First, only single phase devices are being analyzed. Later, the theory will be extended to three phase configurations. The single phase description is valid for decoupled modal components of three phase systems as well.

For building a network equation system, the external behavior of power systems elements is the primary interest. The elements can be partitioned into *inductive* or L-elements, *capacitive* or C-elements and *resistive* or R-elements. The internal behavior is considered by using sources in the external equations. Sources can represent the influence of internal state variables or current or voltage sources.

In this paper, only slow internal state variables like rotor fluxes of machines or mechanical variables are considered which can be evaluated using explicit integration formulas, and hence be treated as sources in the external equations.

In case of fast internal state variables which have to be solved by an implicit method as well, the following equation systems have to be completed by these variables and their equations.

Table I shows the equations of single phase elements, ordered by voltage- and current equations. The following analogy between the equations of L- and C-elements can be observed:

Voltage equations of L-elements and current equations of C-elements are *differential* equations whereas current equations of L-elements and voltage equations of C-elements are *integral* equations. Resistive elements (index R) can be considered as a special case of L- or C-elements (C or $L=0$).

The differential equations of table I are given in the *implicit* state form. The general form of both, L- and C- equations is:

$$x(t) = F\dot{z}(t) + Hz(t) \tag{1}$$

For the transformation into an integral equation, (1) is solved for $\dot{z}(t)$:

$$\dot{z}(t) = -F^{-1}Hz(t) + F^{-1}x(t) = Az(t) + Bx(t) \tag{2}$$

Equation (2) is the *explicit* form of the differential equation. Integrating (2) leads to the integral equations of table I.

The relationship between the coefficients of L- and C elements is:

$$B_L = L^{-1} \qquad A_L = -L^{-1}R_L = -B_L R_L$$
$$B_C = C^{-1} \qquad A_C = -C^{-1}G_C = -B_C G_C$$
$$G_R = R_R^{-1}$$

TABLE I Continuous time equations of power systems elements

Voltage Equation	Current Equation
$u_L(t) = L\dot{i}_L(t) + R_L i_L(t) + u_{qL}(t)$	$i_L(t) = \int_0^t \{A_L i_L(\tau) + B_L [u_L(\tau) - u_{qL}(\tau)]\} d\tau + I_L$
$u_C(t) = \int_0^t \{A_C u_C(\tau) + B_C [i_C(\tau) - i_{qC}(\tau)]\} d\tau + U_C$	$i_C(t) = C\dot{u}_C(t) + G u_C(t) + i_{qC}(t)$
$u_R(t) = R_R i_R(t) + u_{qR}(t)$	$i_R(t) = G_R u_R(t) + i_{qR}(t)$

TABLE II Coefficients of implicit multi-step methods

Method	Ω	History Values $h(k)$
Implicit Euler	$\frac{1}{\Delta t}$	$z(k-1)$
Trapezoidal	$\frac{2}{\Delta t}$	$\Omega^{-1}\dot{z}(k-1) + z(k-1)$
Method acc. to [2]	$\left[A^{-1} - \Delta t \left(e^{A\Delta t} - 1\right)^{-1}\right]^{-1}$	$\left(\Delta t - \Omega^{-1}\right)\dot{z}(k-1) + z(k-1)$

III. Discretization Methods

Difference equations are obtained by discretizing continuous time equations. The discretization can be performed by using either the differential or the integral form of the original continuous time equation. Starting from the differential equation, the operator d/dt has to be approximated by a numerical *differentiator*. Performing the discretization based on the integral equation, the integral $\int \dot{z} dt$ has to be approximated using a numerical *integrator*.

For numerical integrators, explicit as well as implicit approaches can be used. For the discretization with numerical differentiators however, only implicit methods can be applied.

Numerical differentiators are obtained by the following general approach:

$$\dot{z}(k) = D\{z(k), z(k-1), \ldots, z(k-m), \dot{z}(k-1), \ldots, \dot{z}(k-n)\} \quad (3)$$

Numerical integrators are of the following form:

$$z(k) = I\{z(k-1), \ldots, z(k-m), \dot{z}(k), \ldots, \dot{z}(k-n)\} \quad (4)$$

Implicit, linear multi step methods are used very widely. They can be written in the following form, either as numerical differentiator or as numerical integrator:

$$\dot{z}(k) = \Omega z(k) - \Omega h(k) \quad (5)$$

$$z(k) = \Omega^{-1}\dot{z}(k) + h(k) \quad (6)$$

Here, $h(k)$ is the so called history value which comprises values of z and $\dot{z}$ already known at the moment of interest $t = k\Delta t$.

Some numerical integration and differentiation methods, which are often used for the simulation of power systems transients are shown in table II.

IV. Difference Equations of Power Systems Elements

Applying the numerical differentiator (5) to the implicit differential equation (1) the following implicit difference equation is obtained:

$$x(k) = (H + F\Omega)z(k) - F\Omega h(k) \quad (7)$$

Solving for $z(k)$ leads to:

$$z(k) = (H + F\Omega)^{-1}x(k) + (H + F\Omega)^{-1}F\Omega h(k) \quad (8)$$

Using a numerical integrator (6) for discretizing the explicit equation (2), an explicit difference equation results:

$$z(k) = -(A - \Omega)^{-1}Bx(k) - (A - \Omega)^{-1}\Omega h(k) \quad (9)$$

Solving for the input variable $x(t)$ leads to:

$$x(k) = -B^{-1}(A - \Omega)z(k) - B^{-1}\Omega h(k) \quad (10)$$

With:

$$(H + F\Omega) = -B^{-1}(A - \Omega)$$

Equation (7) is equivalent to (10) and equation (8) to (9).

The discrete time equations shown in table III can be obtained accordingly. These equations have the fol-

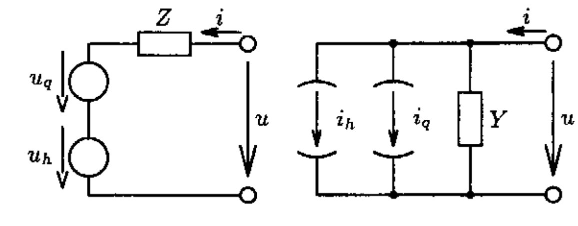

Fig. 1. Equivalent voltage- and current source representation

TABLE III Discrete time equations of power systems elements

Voltage Equation	Current Equation
$u_L(k) = (R_L + L\Omega)i_L(k) + u_{qL}(k) - L\Omega h_L$	$i_L(k) = -(A_L - \Omega)^{-1}B_L u_L(k) + (A_L - \Omega)^{-1}B_L u_{qL}(k) - (A_L - \Omega)^{-1}\Omega h_L(k)$
$u_C(k) = -(A_C - \Omega)^{-1}B_C i_C(k) + (A_C - \Omega)^{-1}B_C i_{qC}(k) - (A_C - \Omega)^{-1}\Omega h_C(k)$	$i_C(k) = (G_C + C\Omega)u_C(k) + i_{qC}(k) - C\Omega h_C(k)$
$u_R(k) = R_R i_R(k) + u_{qR}(k)$	$i_R(k) = G_R u_R(k) + i_{qR}(k)$

TABLE IV Parameter, source and history terms of L- and C-elements

	Z	u_q	u_h	Y	i_q	i_h
L-elm.	$R_L + L\Omega$	$\underline{u_{qL}}$	$\underline{-L\Omega h_L}$	Z_L^{-1}	$-Y_L u_{qL}$	$-Y_L u_{vL}$
C-elm.	Y_C^{-1}	$-Z_C i_{qC}$	$-Z_C i_{hC}$	$G_C + C\Omega$	$\underline{i_{qC}}$	$\underline{-C\Omega h_C}$

lowing common forms for all power systems elements:

$$u(k) = Zi(k) + u_q(k) + u_h(k) \quad (11)$$
$$i(k) = Yu(k) + i_q(k) + i_h(k) \quad (12)$$

As all variables in these equations depend on the same time index k, they can also be written without any time index:

$$u = Zi + u_q + u_h \quad (13)$$
$$i = Yu + i_q + i_h \quad (14)$$

Obviously, current and voltage equations are equivalent considering:

$$Y = Z^{-1} \qquad i_q = -Yu_q \qquad i_h = -Yu_h$$

Table IV shows all expressions for Y and Z and the formulas for the history values i_h and u_h.

The equivalent circuits corresponding to equation (13) and (14) respectively are shown in figure 1.

V. Difference Equations of the Electrical Power System

There are different possibilities for combining the difference equations of power systems elements to a difference equation system describing the complete electrical network. Of particular interest are here methods which are based on Kirchoffs current law. These methods allow to build the complete system in a straight forward way without the necessity of analyzing the topology of the network. The resulting system matrices of these methods are sparse.

There are principally two different methods based on Kirchoffs current law:

- The element current - nodal voltage (EC-NV) approach
- The pure nodal voltage (NV) approach

The well known *difference conductance* method of Dommel [1] ist based on the NV-approach.

A combination between these two systems is the *modified nodal voltage* (MNV) approach [3] which is today the most commonly used method in the field of electronic circuits [4].

For the following derivations, the current law is written using the *node-element-incidence matrix*[1]

$$\boldsymbol{K}\boldsymbol{i}_E = \boldsymbol{0} \quad (15)$$

The complete topology is defined by the incidence matrix $\boldsymbol{K}$. It defines which element is connected to which node. The current vector $\boldsymbol{i}_E$ contains the terminal currents of all elements.

Node voltages and terminal voltages of power systems elements are related as follows:

$$\boldsymbol{u}_E = \boldsymbol{K}^T\boldsymbol{u}_N \quad (16)$$

A. Element Current - Nodal Voltage System

As shown, power systems elements can either be modelled by a current equation according to (14) or by a voltage equation according to (13). Using matrices and vectors, the voltage and current equations (Z- and Y-equations) of all elements can be described as follows:

$$\boldsymbol{Z}\boldsymbol{i}_Z - \boldsymbol{u}_Z = -\boldsymbol{u}_{qZ} - \boldsymbol{u}_{hZ} \quad (17)$$
$$\boldsymbol{i}_Y - \boldsymbol{Y}\boldsymbol{u}_Y = \boldsymbol{i}_{qY} + \boldsymbol{i}_{hY} \quad (18)$$

[1] the node-element incidence matrix corresponds to the node-branch incidence matrix in network theory

The topological equations (15) and (16) can be reordered according to Z- and Y-models:

$$[\boldsymbol{K}_Z \quad \boldsymbol{K}_Y]\begin{bmatrix}\boldsymbol{i}_Z\\ \boldsymbol{i}_Y\end{bmatrix}=0 \tag{19}$$

$$\begin{bmatrix}\boldsymbol{u}_Z\\ \boldsymbol{u}_Y\end{bmatrix}=\begin{bmatrix}\boldsymbol{K}_Z^T\\ \boldsymbol{K}_Y^T\end{bmatrix}\boldsymbol{u}_N \tag{20}$$

Substituting $\boldsymbol{u}_Z$ in (17) and $\boldsymbol{u}_Y$ in (18) by the nodal voltages $\boldsymbol{u}_N$ using (16) results, together with (19), in the EC-NV system:

$$\begin{bmatrix}\boldsymbol{Z} & 0 & -\boldsymbol{K}_Z^T\\ 0 & \boldsymbol{E} & -\boldsymbol{Y}\boldsymbol{K}_Y^T\\ \boldsymbol{K}_Z & \boldsymbol{K}_Y & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{i}_Z\\ \boldsymbol{i}_Y\\ \boldsymbol{u}_N\end{bmatrix}=\begin{bmatrix}-\boldsymbol{u}_{qZ}\\ \boldsymbol{i}_{qY}\\ 0\end{bmatrix}+\begin{bmatrix}-\boldsymbol{u}_{hZ}\\ \boldsymbol{i}_{hY}\\ 0\end{bmatrix} \tag{21}$$

The system matrix of the EC-NV system is sparse and the vector of unknowns is of the dimension $n_{EC-NV} = n_N + n_E$ (number of nodes plus number of elements).

B. Nodal Voltage System

The nodal voltage approach is based on a current source representation for all power system elements according to (14).

Hence, sustituting $\boldsymbol{u}_Y$ in (18) by the current law (15) leads to:

$$\boldsymbol{i}_E - \boldsymbol{Y}\boldsymbol{K}^T\boldsymbol{u}_N = \boldsymbol{i}_q + \boldsymbol{i}_h \tag{22}$$

Multiplying (22) by $\boldsymbol{K}$ and reordering the resulting matrix equation leads to the well known nodal voltage system:

$$\underbrace{\boldsymbol{K}\boldsymbol{Y}\boldsymbol{K}^T}_{\boldsymbol{Y}_N}\boldsymbol{u}_N + \boldsymbol{K}\boldsymbol{i}_q + \boldsymbol{K}\boldsymbol{i}_h = \boldsymbol{K}\boldsymbol{i}_E = 0 \tag{23}$$

Or:

$$\boldsymbol{Y}_N\boldsymbol{u}_N = -\boldsymbol{K}\boldsymbol{i}_q - \boldsymbol{K}\boldsymbol{i}_h \tag{24}$$

The matrix $\boldsymbol{Y}_N$ is the *node-conductance* matrix and can be built directly out of the network structure, analogously to the complex *node-admittance* matrix used for steady state power systems analysis.

The node conductance matrix $\boldsymbol{Y}_N$ is reflecting the network topology. It is sparse and symmetrical, like the node admittance matrix, a fact from which equation solvers can highly benefit.

The dimension of $\boldsymbol{Y}_N$ is $n_{NV} = n_N$ (number of nodes). $\boldsymbol{Y}_N$ is therefore much smaller than the system matrix of the EC-NV system.

But, as every element has to be modelled by a current equivalent (Y-equation), the node conductance matrix can get badly conditioned in case of very small difference resistances Z as they can result fro C-elements together with very small integration step sizes (see also table IV).

C. Modified Nodal Voltage System

The modified nodal voltage approach combines the advantages of the EC-NV approach and the pure NV approach.

Analogously to the the pure nodal voltage approach, $\boldsymbol{u}_Y$ of the current sources-equations (18) is substituted by $\boldsymbol{K}_Y^T\boldsymbol{u}_N$. Multiplying equation (18) by $\boldsymbol{K}_Y$ results in:

$$\boldsymbol{K}_Y\boldsymbol{i}_Y - \underbrace{\boldsymbol{K}_Y\boldsymbol{Y}\boldsymbol{K}_Y^T}_{\boldsymbol{Y}_{NY}}\boldsymbol{u}_N = \boldsymbol{K}_Y\boldsymbol{i}_{qY} + \boldsymbol{K}_Y\boldsymbol{i}_{hY} \tag{25}$$

Using (19) in the form:

$$\boldsymbol{K}_Z\boldsymbol{i}_Z + \boldsymbol{K}_Y\boldsymbol{i}_Y = 0$$

the element currents of Y-elements $\boldsymbol{i}_Y$ can be eliminated from the vector of unknowns:

$$\boldsymbol{K}_Z\boldsymbol{i}_Z + \boldsymbol{Y}_{NY}\boldsymbol{u}_N = -\boldsymbol{K}_Y\boldsymbol{i}_{qY} - \boldsymbol{K}_Y\boldsymbol{i}_{vY} \tag{26}$$

The voltage source equation (19) together with (26) are building the modified nodal voltage (MNV) system:

$$\begin{bmatrix}\boldsymbol{Z} & -\boldsymbol{K}_Z^T\\ \boldsymbol{K}_Z & \boldsymbol{Y}_{NY}\end{bmatrix}\begin{bmatrix}\boldsymbol{i}_Z\\ \boldsymbol{u}_N\end{bmatrix}=\begin{bmatrix}-\boldsymbol{u}_{qZ}\\ -\boldsymbol{K}_Y\boldsymbol{i}_{qY}\end{bmatrix}+\begin{bmatrix}-\boldsymbol{u}_{hZ}\\ -\boldsymbol{K}_Y\boldsymbol{i}_{hY}\end{bmatrix} \tag{27}$$

The dimension of the system matrix is here $n_{MNV} = n_Z + n_N$ (number of Z-elements plus number of nodes). The MNV-system is much smaller than the EC-NV system, but it can still represent voltage equivalents (Z-equations) as well. Therefore, numerical problems in case of very small integration step sizes can be avoided.

The flexibility in modelling power systems elements either by current- or by voltage equations can also be used for transforming the implicit equation system to a state space form, as shown in the next section.

D. Explicit State Space Representation

The representation of difference equations with history values on the right side is valid for all discretization methods. This is not the case if the system is transformed into an explicit state space form.

The implicit formulation according to the previous sections is very efficient for a recursive solution of the difference equation system. The explicit state space representation however can be used for analyzing the system in a more analytical way. Analogously to a continuous time state space representation, the discrete time system allows to analyze stability properties by calculating eigenvalues and eigenvectors. Also characteristic frequencies can directly be found by analyzing the calculated eigenvalues of the discrete time system (see e.g. [5]).

In real time applications, in which a linear electrical grid can be assumed[2], the explicit formulation allows a direct solution of the system allowing to 'skip' some time steps without increasing the discretization error which leads to very fast simulation algorithms.

The difficulty of transforming the implicit formulation according to (27) into an explicit state space system depends highly on the decision which element is

[2] the assumption $x_d'' = x_q''$ is also necessary

TABLE V Single phase and three phase systems

	single phase	three phase	
		abc-co-ordinates	modal co-ordinates
Variable/vectors	g	$g = \begin{bmatrix} g_a & g_b & g_c \end{bmatrix}^T$	$g_M = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}^T$
Parameter/matrices	M	$M = \begin{bmatrix} M_{aa} & M_{ab} & M_{ac} \\ M_{ba} & M_{bb} & M_{bc} \\ M_{ca} & M_{cb} & M_{cc} \end{bmatrix}$	$M_M = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}$

modelled as a voltage (Z) and which element as a current (Y) equation.

If every L-element is modelled as voltage equivalent (Z-equation) and every C-element as current equivalent (Y-equation) the transformation can be performed without any difficulties.

Modelling power systems elements by this way, the vector of unknowns comprises all inductive currents and all node voltages, hence the storage variables are building a subset of the vector of unknowns.

The modified nodal voltage system can then be expressed as follows:

$$\begin{bmatrix} Z_L & -K_L^T \\ K_L & Y_{NC} \end{bmatrix} \begin{bmatrix} i_L \\ u_N \end{bmatrix} = \begin{bmatrix} -u_{qL} \\ -K_C i_{qC} \end{bmatrix} + \begin{bmatrix} -u_{hL} \\ -K_C i_{hC} \end{bmatrix} \tag{28}$$

For building the state space system out of (28), the history terms u_{hL} and i_{hC} have to be expressed by i_L, u_N, u_{qL} and i_{qC} at the previous time step $k-1$.

Using trapezoidal rule, the history values $u_{hL}(k)$ and $i_{hC}(k)$ can be expressed as follows:

$$\begin{aligned} u_{hL}(k) &= -L\Omega h_L(k) \\ &= -L\Omega\left[\Omega^{-1}\dot{i}_L(k-1) + i_L(k-1)\right] \\ &= (R_L - L\Omega) i_L(k-1) - u_L(k-1) + u_{qL}(k-1) \end{aligned} \tag{29}$$

and

$$\begin{aligned} i_{hC}(k) &= -C\Omega h_L(k) \\ &= -C\Omega\left[\Omega^{-1}\dot{u}_C(k-1) + u_C(k-1)\right] \\ &= (G_C - C\Omega) u_C(k-1) - i_C(k-1) + i_{qC}(k-1) \end{aligned} \tag{30}$$

Using the abbreviations

$$Z_L^* = R_L - L\Omega \tag{31}$$

$$Y_C^* = G_C - C\Omega \tag{32}$$

the history terms of all elements are:

$$u_{hL}(k) = Z_L^* i_L(k-1) - u_L(k-1) + u_{qL}(k-1) \tag{33}$$

$$i_{hC}(k) = Y_C^* u_C(k-1) - i_C(k-1) + i_{qC}(k-1) \tag{34}$$

Finally, the element voltages $u_L(k-1)$ and $u_C(k-1)$ have to be substituted by $u_N(k-1)$ using (20). The capacitive element currents $i_C(k-1)$ must be eliminated as well. This can be done by applying Kirchoffs current law according to (19) again:

$$K_C i_C(k-1) = K_L i_L(k-1)$$

The history terms (33) and (34) can then be rewritten:

$$u_{hL}(k) = Z_L^* i_L(k-1) - K_L^T u_N(k-1) + u_{qL}(k-1) \tag{35}$$

$$K_C i_{hC}(k) = \underbrace{K_C Y_C^* K_C^T}_{Y_{NC}^*} u_N(k-1) + K_L i_L(k-1) + K_C i_{qC}(k-1) \tag{36}$$

Replacing the history terms in (28) by (34) and (33) and inverting the system matrix the electrical network can be described in a discrete time state space:

$$\begin{aligned} \begin{bmatrix} i_L(k) \\ u_N(k) \end{bmatrix} = &\begin{bmatrix} Z_L & -K_L^T \\ K_L & Y_{NC} \end{bmatrix}^{-1} \begin{bmatrix} -Z_L^* & K_L^T \\ -K_L & -Y_{NC}^* \end{bmatrix} \begin{bmatrix} i_L(k-1) \\ u_N(k-1) \end{bmatrix} \\ &- \begin{bmatrix} Z_L & -K_L^T \\ K_L & Y_{NC} \end{bmatrix}^{-1} \begin{bmatrix} u_{qL}(k) + u_{qL}(k-1) \\ K_C i_{qC}(k) + K_C i_{qC}(k-1) \end{bmatrix} \end{aligned} \tag{37}$$

VI. Three Phase Systems

The transfer from single phase to three phase systems can be performed very easily by replacing each variable by a vector and each parameter by a matrix (see. table V).

The description by parameter matrices is valid for a representation in original (abc) as well as in modal co-ordinate systems.

In the derivation of single phase equation systems, divisions have been avoided and the inversion operator $(\ldots)^{-1}$ has been used in the right sequence instead, so that the replacement of scalar parameters by matrices can be performed in a purely formal way. Additionally, the unit matrix has to be used in the incidence matrices instead of the scalar 1,

VII. Example

The example network of figure 2 consists of three L-elements, one C-element and one R-element.

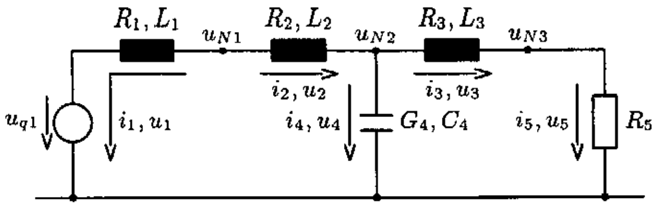

Fig. 2. Example configuration

As in the previous sections, the L-elements are modelled by voltage (Z) equations, the C- and the R-element by current (Y) equations.

With the indices from figure 2, the following equation systems are resulting:
Element current - nodal voltage (EC-NV) system according to (21):

$$\begin{bmatrix} Z_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & Z_2 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & Z_3 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -Y_4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -Y_5 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ u_{N1} \\ u_{N2} \\ u_{N3} \end{bmatrix} = \begin{bmatrix} -u_{q1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -u_{h1} \\ -u_{h2} \\ -u_{h3} \\ i_{h4} \\ i_{h5} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (38)$$

Nodal voltage (NV) system according to (24):

$$\begin{bmatrix} Y_1+Y_2 & -Y_2 & 0 \\ -Y_2 & Y_2+Y_3+Y_4 & -Y_3 \\ 0 & -Y_3 & Y_3+Y_5 \end{bmatrix} \begin{bmatrix} u_{N1} \\ u_{N2} \\ u_{N3} \end{bmatrix} = \begin{bmatrix} -i_{q1} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -i_{h1}-i_{h2} \\ i_{h2}-i_{h3}-i_{h4} \\ i_{h3}-i_{h5} \end{bmatrix} \quad (39)$$

Modified nodal voltage system according to (27):

$$\begin{bmatrix} Z_1 & 0 & 0 & -1 & 0 & 0 \\ 0 & Z_2 & 0 & -1 & 1 & 0 \\ 0 & 0 & Z_3 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & Y_4 & 0 \\ 0 & 0 & -1 & 0 & 0 & Y_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_{N1} \\ u_{N2} \\ u_{N3} \end{bmatrix} = \begin{bmatrix} -u_{q1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -u_{h1} \\ -u_{h2} \\ -u_{h3} \\ 0 \\ -i_{h4} \\ -i_{h5} \end{bmatrix} \quad (40)$$

State space form according to (37):

$$\begin{bmatrix} Z_1 & 0 & 0 & -1 & 0 & 0 \\ 0 & Z_2 & 0 & -1 & 1 & 0 \\ 0 & 0 & Z_3 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & Y_4 & 0 \\ 0 & 0 & -1 & 0 & 0 & Y_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_{N1} \\ u_{N2} \\ u_{N3} \end{bmatrix}_{(k)} = - \begin{bmatrix} Z_1^* & 0 & 0 & -1 & 0 & 0 \\ 0 & Z_2^* & 0 & -1 & 1 & 0 \\ 0 & 0 & Z_3^* & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & Y_4^* & 0 \\ 0 & 0 & -1 & 0 & 0 & Y_5^* \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_{N1} \\ u_{N2} \\ u_{N3} \end{bmatrix}_{(k-1)} - \begin{bmatrix} u_{q1}(k)+u_{q1}(k-1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (41)$$

VIII. Concluding Remarks

In contrast to the classical nodal voltage system, not only current but also voltage equivalents can be used together with the EC-NV- or the MNV-system. Hence, a high flexibility in modelling power systems elements is offered leading to equation systems with less numerical problems even in case of very small integration step sizes.

Finally, it has been shown, how the modified nodal voltage system can be transformed easily into a discrete time state space formulation to which the complete theory of discrete time systems can be applied. The explicit form can also be used for very fast simulation algorithms as required by real time applications.

The described methods have been implemented and successfully tested in the power systems analysis software DIgSILENT *PowerFactory*.

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