Accuracy of Discretization Methods for Electromagnetic Transient Simulation

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Abstract **– Electromagnetic transient simulation has become a very important tool in the design of electrical power systems. The method of substituting the trapezoidal integrator, developed by Dommel, transform the differential equation into a difference equation has become the standard method for electromagnetic transient simulations. The trapezoidal rule is based on a truncated Taylor series and therefore contains truncation errors. These truncation errors cause numerical oscillations when the time step is large relative to some of the time constants in the network. In this paper the performance of various methods for developing difference equations are compared for a number of input waveforms.**

Keywords – Simulation techniques, error, integration techniques, root-matching

I. INTRODUCTION

The continuous nature of the power system's dynamic behaviour can be represented mathematically by differential equations, however, digital simulation on the other hand is a discrete process. The main simulation task is, therefore, to find a method for determining the solution of the differential equations representing the power system at discrete time points. This can be efficiently achieved through the use of difference equations. The method of substituting the trapezoidal integrator into the differential equation, developed by Dommel, is one method for developing a difference equation that simulates the differential equation [1]. Any other numerical integration formulae can be used. An alternative to numerical integrator substitution (NIS) is the use of the root-matching technique to develop an exponential difference equation. Unlike the former the later is applied to series or parallel RL, RC, LC and RLC combinations not individual R,L or C components. This paper presents a study of the error characteristics of numerical integrator substitution method using various integrators as well as root-matching techniques. Simulation error as a function of time-step length and type of input function is investigated, using a simple RL circuit.

II. DIFFERENCE EQUATION

Regardless of the technique employed, the process of Discretization (developing a discrete representation of and continuous process) results in a difference equation being generated from the differential equations, that has the form:

$$\begin{aligned} y(t) &= a_0 u(t) \\ &+ a_1 u(t-\Delta t) + a_2 u(t-2\Delta t) + \cdots a_N u(t-N\Delta t) \\ &- b_1 y(t-\Delta t) - b_2 y(t-2\Delta t) - \cdots b_n y(t-n\Delta t) \end{aligned} \quad (1)$$

Where u is the input and y the output. When used to represent electrical components the voltage is the input and current the output quantity. Equation 1 can also be viewed as a Norton equivalent where the conductance is a_0 and the remainder a history term, as shown in Fig. 1. Taking the z-transform and rearranging gives the transfer function:

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots a_N z^{-N}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots b_n z^{-n}} \quad (2)$$

Each digital implementation can therefore be viewed from its z-domain representation (transfer function) or equivalent difference equation. Stability of the difference equation can be determined by the positions of the poles of $H(z)$.

III. DEVELOPMENT OF DIFFERENCE EQUATIONS

Dommel's method is an example of Numerical Integrator Substitution (NIS) method, which uses Trapezoidal integrator. Any other numerical integrator can be substituted into the difference equation to form a difference equation for simulation [2]. Out of the numerous possible integration formulae the ones considered in this paper are:

Gear 2nd Order $y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2\Delta t}{3} f_{n+1}$ (3)

Backward Euler $y_{n+1} = y_n + \Delta t f_{n+1}$ (4)

Forward Euler $y_{n+1} = y_n + \Delta t f_n$ (5)

And of course the

Trapezoidal rule $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f_{n+1} + f_n\right)$ (6)

The Trapezoidal integrator is one of a family of integra-

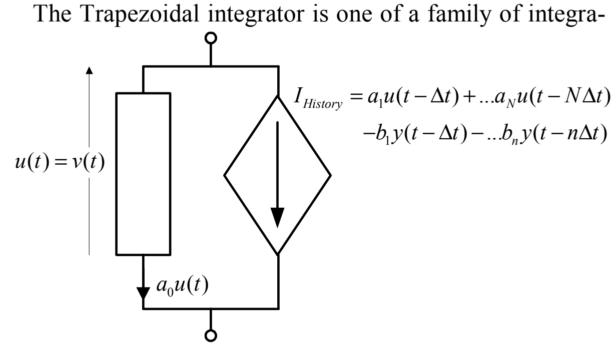

Fig. 1 Norton Equivalent

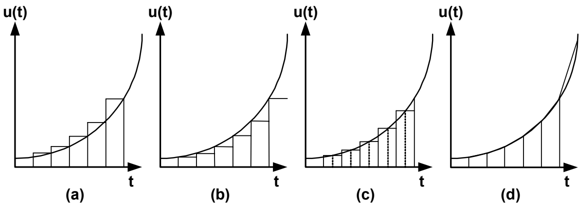

Fig. 2. Family of Root-Matching

tion rules that can be obtained from the Tunable integrator $y_{n+1} = y_n + \lambda \Delta t \left(\gamma f_{n+1} + (1-\gamma) f_n \right)$ where the gain parameter $\lambda = 1$ and phase parameter $\gamma = 1/2$ [2].

Substitution of the Trapezoidal rule into the differential equation for an RL circuit leads to:

$$i(t) = \frac{\left(1 - \frac{\Delta t R}{2L}\right)}{\left(1 + \frac{\Delta t R}{2L}\right)} i(t-\Delta t) + \frac{\left(\frac{\Delta t}{2L}\right)}{\left(1 + \frac{\Delta t R}{2L}\right)} \left(v(t) + v(t-\Delta t) \right)$$

$$= \underbrace{Gv(t)}_{\text{Instant}} + \underbrace{k_v v(t-\Delta t) + k_i i(t-\Delta t)}_{\text{History}} \quad (7)$$

An alternative to NIS is the use of root-matching technique [3]. The aim is to match the dynamics of the difference equation with the dynamics of the differential equations it represents. To achieve this the root-matching method matches the positions of the poles and zeros of the simulation equation to the continuous equation it represents. There is however a family of root-matching formula depending on the assumed variation in input between time points (shown in Fig. 2) [4]. For example consider Root-matching for a simple RL branch with type (a) input. The Transfer function (continuous) is:

$$H(s) = \frac{G}{1+s\tau} = \frac{1/R}{1+s\,L/R} \quad (8)$$

The discrete transfer function is: $\mathrm{H(z)} = \dfrac{K}{1 - z^{-1} e^{-\Delta t/\tau}}$ (9)

Using the final value theorem for continuous and discrete systems and equating i.e.

$$\mathrm{Lim}_{s \to 0} \left\{ sH(s)/s \right\} = 1/R \quad (10)$$

$$\mathrm{Lim}_{z \to 1} \left\{ \frac{(z\text{ -}1)}{z} H(z) \Big/ \frac{z}{(z-1)} \right\} = \frac{K}{\left(1 - e^{-\Delta t/\tau}\right)} \quad (11)$$

equating gives $K = \left(1 - e^{-\Delta t/\tau}\right)/R$ and the corresponding difference equation is therefore:

$$i(t) = \frac{1}{R}(1 - e^{-\Delta t/\tau}) v(t) + e^{-\Delta t/\tau} i(t-\Delta t) \quad (12)$$

This requires less computation than the difference equation for trapezoidal rule (equation (7)) as the exponential terms can be pre-calculated and stored. Only one value from previous time-point is required with two multiplications and one addition are required at each time-point. Application of this to an RL circuit for different members of the root-matching family gives:

Input type (a): $$\frac{I(z)}{V(z)} = \frac{\frac{1}{R}\left(1 - e^{-\Delta t/\tau}\right)}{\left(1 - z^{-1} e^{-\Delta t/\tau}\right)} \quad (13)$$

Input type (b): $$\frac{I(z)}{V(z)} = \frac{\frac{1}{R}\left(1 - e^{-\Delta t/\tau}\right) z^{-1}}{\left(1 - z^{-1} e^{-\Delta t/\tau}\right)} \quad (14)$$

Input type (c): $$\frac{I(z)}{V(z)} = \frac{\left(1 - e^{-\Delta t/\tau}\right)\left(1 + z^{-1}\right)}{2R\left(1 - z^{-1} e^{-\Delta t/\tau}\right)} \quad (15)$$

Input type (d):

$$\frac{I(z)}{V(z)} = \frac{\left[-e^{-a\Delta t/\tau} + \frac{\tau}{\Delta t}\left(1 - e^{-a\Delta t/\tau}\right) \right] z^{-1} + \frac{1}{R}\left[1 - \frac{\tau}{\Delta t}\left(1 - e^{-a\Delta t/\tau}\right) \right]}{R\left(1 - z^{-1} e^{-a\Delta t/\tau}\right)} \quad (16)$$

Another method of generating the difference equation is the direct optimization of the coefficients to minimize the error in the frequency domain [5,6]. Although this works well, optimisation is computationally expensive and needs to be performed every time the step-length is altered.

IV. Simulation Results

A. Step Input

The first test is a simple step response. For an RL circuit the exact response is:

$$i(t) = \frac{V}{R}\left(1 - e^{-(t-t_1)/\tau}\right) \qquad t > t_1 \quad (17)$$

where t_1 is the time the step is applied.

Figures 3 and 4 show the current waveform for step input with simulation step length of 50 and 500 μsec respectively (time constant of RL is 50 μsec). Figure 5 displays the error and exact current as a function of time whereas Tables I to III summarize the maximum error and time that it occurs. Root matching A gives the exact answer regardless of the time-step. Forward Euler and Root-matching B always give the maximum error on the first time point after the change in input (100%) due to the delay in response. Due to the exponential form of the difference equation the error in the root-matching RM-B response decays to zero. RM-C always gives a 50% error at the first time point after the step. The forward Euler method is unstable for a time-step of 500 μsec. In fact the region of its stability is given by $0 < \Delta t/\tau < 2$, hence a time-step of 100 μsec is on the limit giving undamped oscillatory response.

B. Ramp Input

When there is an abrupt change the transient seen by a component remote to the inception is more like a time limited ramp (Fig. 6). Therefore a time limited ramp is the second input test signal. This ramp signal can be considered to be made of a positive ramp starting at time t_1 and negative ramp, with the same magnitude of slope, starting at t_2. The difference between start times being the ramp up time (t_R). Hence

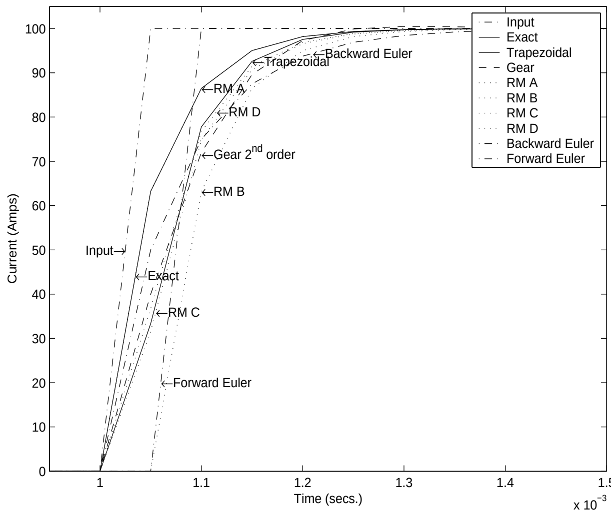

Fig. 3 Step Response with 50 µsec time-step

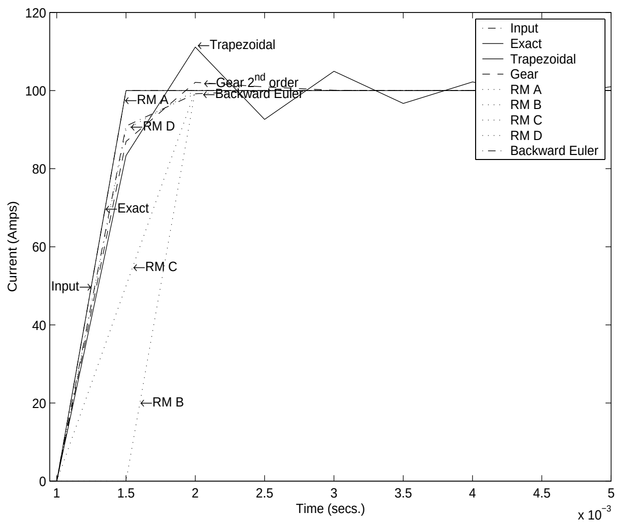

Fig 4 Step Response with 500 µsec time-step

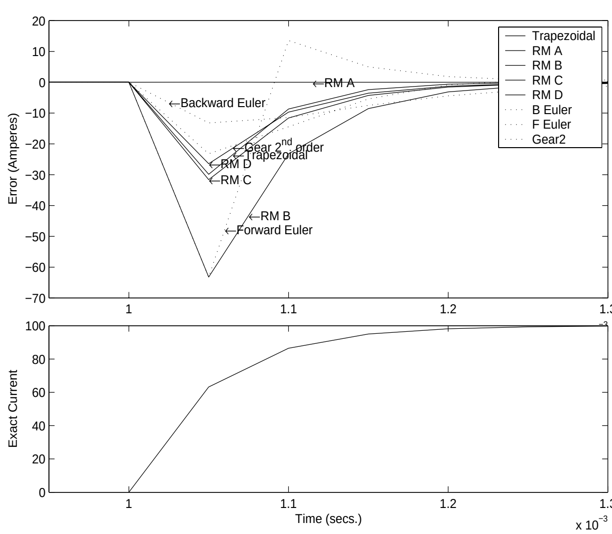

Fig. 5 Step Response with 50 µsec time-step

$$V(s) = \left(e^{-st_1} - e^{-st_2}\right)\frac{k}{s^2} \tag{18}$$

$$I(s) = \frac{1}{R + sL}\left(e^{-st_1} - e^{-st_2}\right)\frac{k}{s^2} \tag{19}$$

where k is the ramp slope (V/t_R)

Expanding this into partial fractions and taking the inverse Laplace transform gives the expression:

$$\begin{aligned} i(t) &= \frac{k}{R}(t - t_1) - \frac{kL}{R^2}\left(1 - e^{-(t-t_1)/\tau}\right) \\ &\quad - \frac{k}{R}(t - t_2) + \frac{kL}{R^2}\left(1 - e^{-(t-t_2)/\tau}\right) \qquad t_2 < t \\ &= \frac{k}{R}(t_2 - t_1) - \frac{kL}{R^2}\left(e^{t_2/\tau} - e^{t_1/\tau}\right)e^{-t/\tau} \end{aligned} \tag{20}$$

When $t_1 < t < t_2$ the current is given by:

$$i(t) = \frac{k}{R}(t - t_1) - \frac{kL}{R^2}\left(1 - e^{-(t-t_1)/\tau}\right) \tag{21}$$

Table I Step Response (5 µsec time-step)

Method	Error		Time	Exact
	(Amps)	(%)	(msec)	(Amps)
Trap.	4.75435	49.960	1.0050	9.516258
B. Euler	1.76639	2.794	1.0500	63.212056
F. Euler	9.51626	100.000	1.0050	9.516258
Gear 2nd	4.06443	22.422	1.0100	18.126925
RM A	0.00000	0.000	1.0050	9.516258
RM B	9.51626	100.000	1.0050	9.516258
RM C	4.75813	50.000	1.0050	9.516258
RM D	4.67884	49.167	1.0050	9.516258

Table II Step Response (50 µsec time-step)

Method	Error		Time	Exact
	(Amps)	(%)	(msec)	(Amps)
Trap.	29.87872	47.267	1.0500	63.21206
B. Euler	13.21206	20.901	1.0500	63.21206
F. Euler	63.21206	100.000	1.0500	63.21206
Gear 2nd	23.21206	36.721	1.0500	63.21206
RM A	0.00000	0.000	1.0500	63.21206
RM B	63.21206	100.000	1.0500	63.21206
RM C	31.60603	50.000	1.0500	63.21206
RM D	26.42411	41.802	1.0500	63.21206

Table III Step Response (500 µsec time-step)

Method	Error		Time	Exact
	(Amps)	(%)	(msec)	(Amps)
Trap.	16.66213	16.663	1.5000	99.99546
B. Euler	9.08637	9.087	1.5000	99.99546
F. Euler	-	-	-	-
Gear 2nd	13.03894	13.040	1.5000	99.99546
RM A	0.00000	0.000	2.5000	100.00
RM B	99.99546	100.000	1.5000	99.99546
RM C	49.99773	50.000	1.5000	99.99546
RM D	9.99501	9.087	1.5000	99.99546

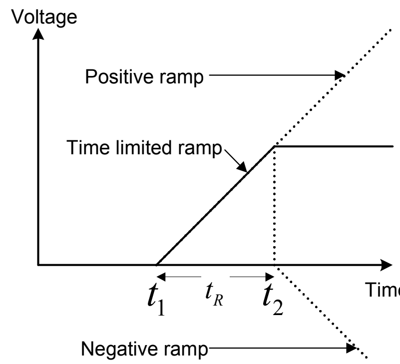

Fig. 6 Time Limited Ramp Input

Figures 7 and 8 show the error for a ramp time of 0.1 and 1 msec, respectively. One noticeable feature is the constant error in RM-A, RM-B and RM-C while ramping up (Fig. 8). Tables IV to VI give the error for various simulation time-step, for a rise time of 1 msec. Comparison of percentage error can be very misleading as high percentage error can be observed when the ramp is just beginning due to the low exact value, hence actual error and place maximum occurs are shown. Fig. 9 displays the error for each technique as a function of time for a number of ramp rates

Table IV 5 µsec time-step (Ramp Time 1 msec).

Method	Error		Time	Exact
	(Amps)	(%)	(msec.)	(Amps)
Trap.	0.00153	0.083	1.0050	1.839397
B. Euler	0.08832	4.802	1.0500	1.839397
F. Euler	0.09601	5.219	1.0500	1.839397
Gear 2nd	0.00791	0.008	2.0100	95.90635
RM A	0.24583	0.259	2.0000	95.00000
RM B	0.25417	0.268	2.0000	95.00000
RM C	0.00417	0.004	2.0000	95.00000
RM D	0.00000	0.000	2.4350	99.99917

Table V 50 µsec time-step (Ramp Time 1 msec).

Method	Error		Time	Exact
	(Amps)	(%)	(msec.)	(Amps)
Trap.	0.17273	9.391	1.0500	1.839397
B. Euler	0.66060	35.914	1.0500	1.839397
F. Euler	1.83940	100.000	1.0500	1.839397
Gear 2nd	0.16894	1.648	1.1500	10.24894
RM A	2.09012	2.200	2.0000	95.00000
RM B	2.90988	3.063	2.0000	95.00000
RM C	0.40988	0.431	2.0000	95.00000
RM D	0.00000	0.000	2.5000	99.99977

Table VI 500 µsec time-step (Ramp Time 1 msec).

Method	Error		Time	Exact
	(Amps)	(%)	(msec.)	(Amps)
Trap.	3.33356	7.408	1.5000	45.00023
B. Euler	0.45432	1.010	1.5000	45.00023
F. Euler	-	-	-	-
Gear 2nd	1.52197	3.382	1.5000	45.00023
RM A	4.99773	5.261	2.0000	95.00000
RM B	45.00227	47.371	2.0000	95.00000
RM C	20.00227	21.055	2.0000	95.00000
RM D	0.00000	0.000	2.0000	95.00000

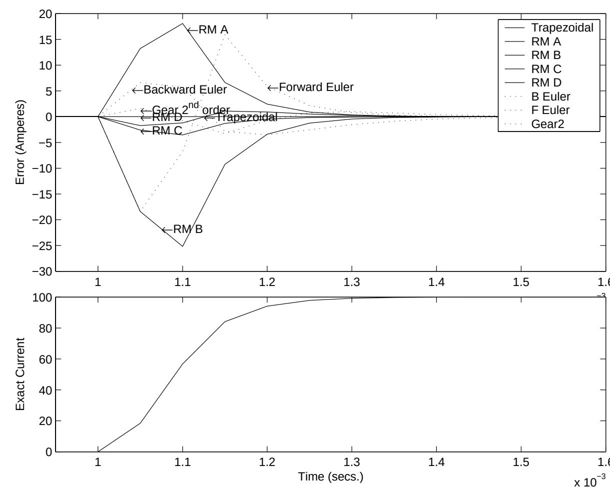

Fig. 7 Error for a 0.1 msec ramp Input

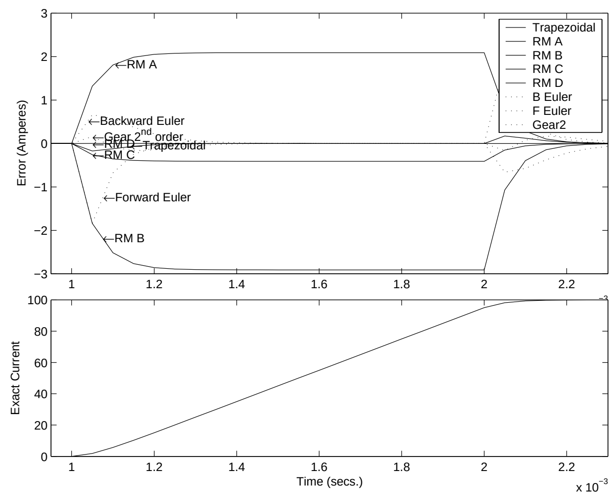

Fig. 8 Error for a 1 msec ramp Input

(ramp times are given in Table VII).

It is clear that for the Trapezoidal integrator the error decreases as the ramp time increases. This is also true for Gear 2nd order, RM-B, RM-C, RM-D and Forward Euler. Gear 2nd order in fact performs marginally better than the Trapezoidal rule. RM-A gives no error for a step input and performs worse as the ramp time increase to a certain point then improves again. The backward Euler performs well for steep input but performs worse than Trapezoidal rule for long ramp times.

C. Switched Sinusoid

An alternating voltage is often applied to a network by closing a switch. Hence:

$$I(s) = \frac{V\omega}{\left(s^2+\omega^2\right)\left(R+sL\right)} \tag{22}$$

Again expanding this into partial fractions and taking the inverse Laplace transform gives the expression:

$$i(t) = \frac{V}{\sqrt{R^2+(\omega L)^2}}\left(e^{-t/\tau}\sin(\phi)+\sin(\omega t-\phi)\right) \tag{23}$$

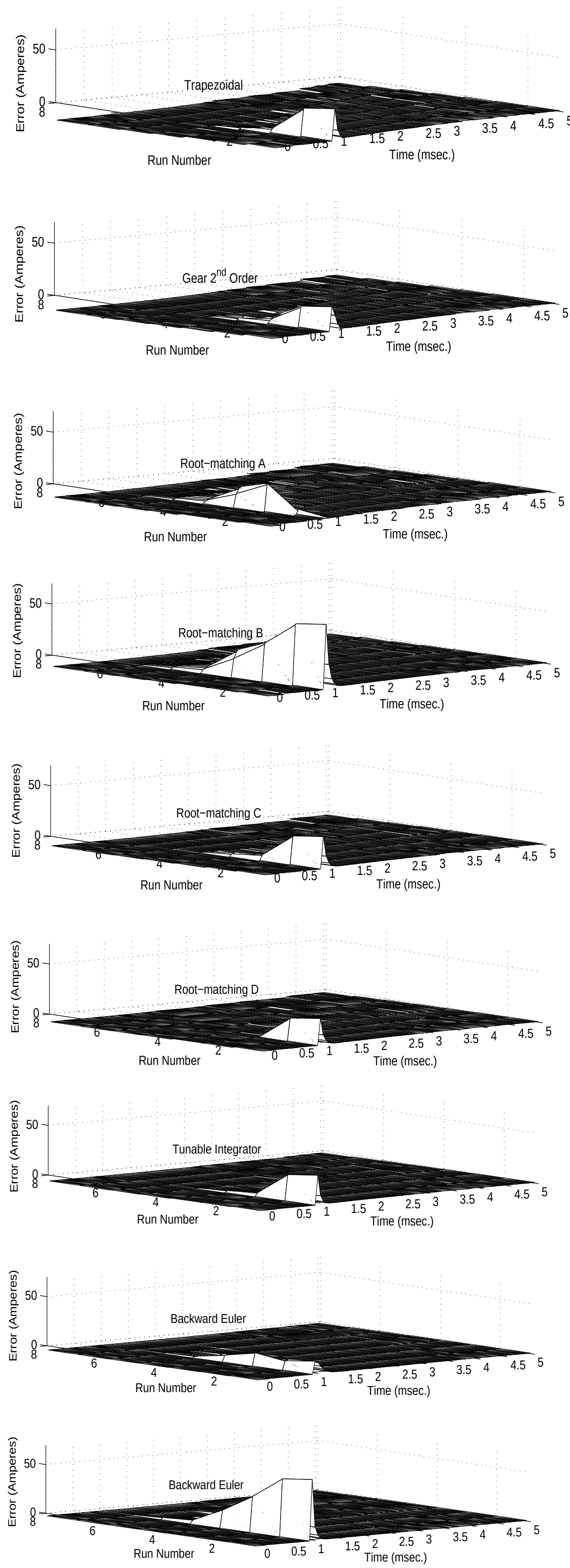

Fig. 9 Three-Dimensional Plot of Error for various ramp rates (50 μsec time-step)

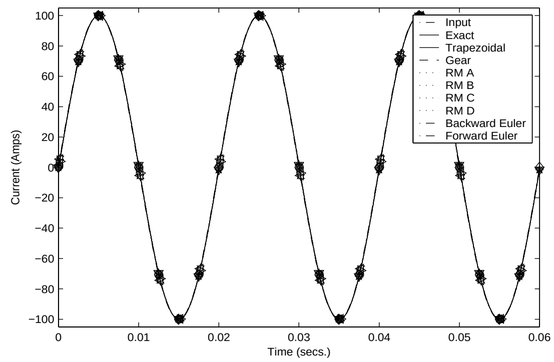

Fig. 10 Simulation results for Switched Sinusoid

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is misleading as high values are registered when exact current is very small. The differences are not perceivable in the waveforms shown in Figure 10. With a time-constant of 50 μsec the transient term is small and hence the peaks all look the same in Figures 10 and 11. By increasing the time constant (increasing *L*) by a factor of 100 the transient term is clearly evident in the simulation results (Fig. 13) as the peaks reduce. The error is displayed in Fig. 14. As there

Table VII Rise times for ramp input

Run No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Rise time (msec)	0.0	0.01	0.05	0.1	0.5	1	5	10

Table VIII 50 μsec time-step (Switched Sinusoid).

Method	**Error** (Amps)	(%)	**Time** (msec)	**Exact** (Amps)
Trap.	0.05427	9.39	0.050	0.57785
B. Euler	0.20752	35.91	0.050	0.57785
F. Euler	0.57785	100.00	0.050	0.57785
Gear 2nd	0.07535	2.34	0.150	3.21900
RM A	0.65655	508332.	20.050	0.00013
RM B	0.91405	707701.	40.050	0.00013
RM C	0.12877	8.20	40.100	1.57067
RM D	0.00202	0.002	0.050	0.57785

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of each technique has stayed the same.

V. CONCLUSIONS

This paper has compared the various techniques for developing difference equations for simulation electromagnetic transients. Comparisons have been made under the arduous conditions of when the step-length is large compared to the circuit time-constant, for a number of different input functions.

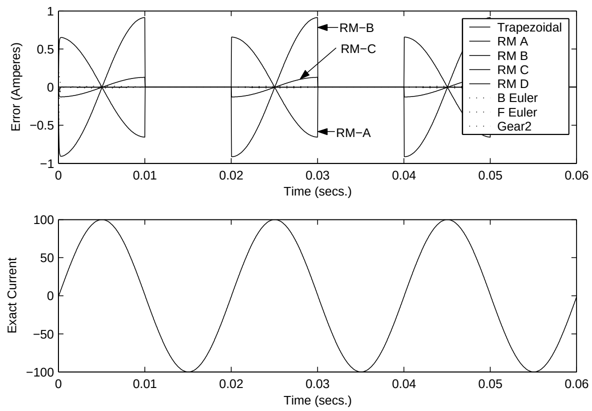

Fig. 11 Error for Switched Sinusoid

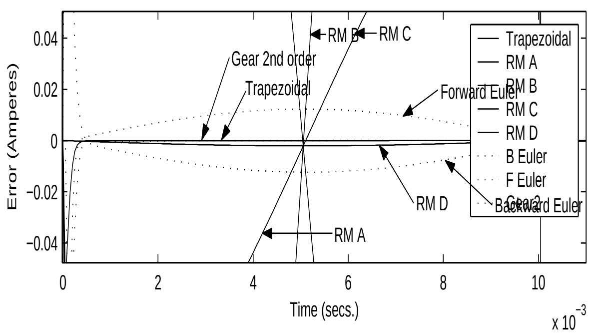

Fig. 12 Zoomed view of Error for Switched Sinusoid

The root-matching technique ensures that a match between the poles and zeros positions of the difference equation match those of the continuous system they represent. However, there is still a supposition of how the input varies between time-points, if this supposition is correct then no error occurs. For example RM-A with step input and RM-D with ramp input. Because of the exponential form of difference equation obtained from root-matching equation the simulation error tends to decay to zero rather than oscillate. The results show there is no merit in RM-B or RM-C for the input functions investigated. As expected from the formulation, the error in RM-C is an average between the error in RM-A and RM-B. The performance of Forward Euler shows why it is an undesirable method. Trapezoidal, Backward Euler and Gear 2nd order all performed satisfactorily. There is no clear winner as each integration method is marginally better than the other for some test input functions.

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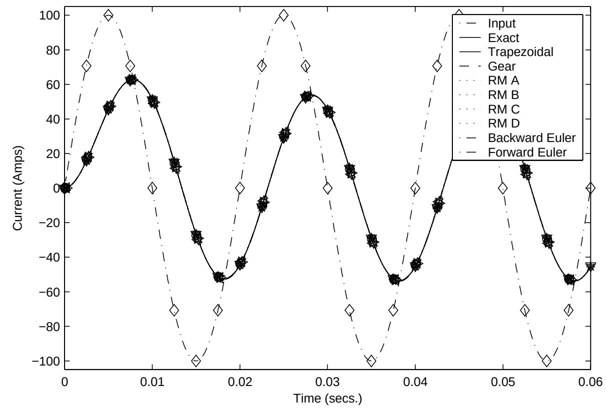

Fig. 13 Switch Sinusoid

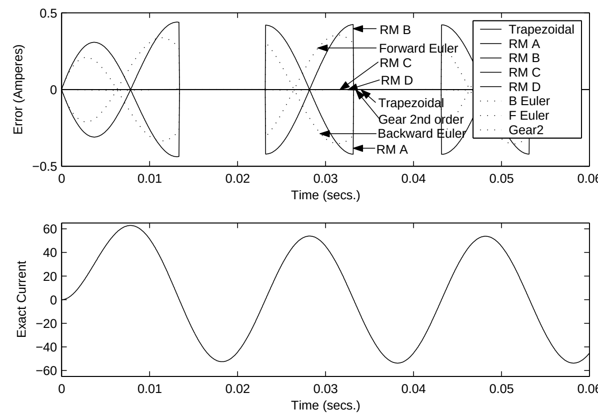

Fig. 14 Error in Switched Sinusoid Simulation

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