Transmission Line Model Based on Vector Fitting and Rank Constrained Minimization

Ramin Mirzahosseini, Reza Iravani

***Abstract*-- There has been several attempts to detailed modeling of the transmission line for the analysis of the electromagnetic transients. One of the most accurate models for the line representation is the universal model. This model is based on the Vector Fitting to obtain a rational approximation of the transmission line model which results in high-order time domain realization. This work proposes a new method, based on the universal model and rank constrained optimization to achieve a low-order realization of the transmission line propagation function with the desired accuracy. This reduces the computation burden by a factor equal to number of conductors. The approach has been applied to an overhead transmission line and the results satisfactorily match with the PSCAD results.**

***Keywords*: transmission line, universal model, rank constrained minimization, minimal realization.**

I. Introduction

TIME-DOMAIN simulation of electromagnetic transients is essential for design, study and planning of power systems. The transmission line model contains two irrational transfer function matrices, i.e., the characteristic admittance matrix *Y(s)* and the propagation transfer function matrix *H(s)*. The line modelling for time-domain simulation requires approximation of these two transfer functions with rational transfer functions which are inherently easier to discretize and integrate.

These approximations are still being investigated to deduce more realistic emulation of the physical behavior of the overhead line and the underground cable. In most works, approximation of *Y(s)* and *H(s)* are done separately and independently. Vector fitting [1] is used in both approximations as the core technique where in case of *Y(s)* the passivity enforcement is also introduced to ensure that the line model is passive [2] [3].

For the approximation of *H(s)*, there are two main approaches, i.e., the modal-domain model and the phase-domain model where the later one is more accurate but computationally more demanding. The differences in these two groups are as follows. Due to the presence of different time delays in the propagation function, it is not possible to apply vector fitting unless these delays are extracted from the propagation function. Modal decomposition is used to decompose the matrix into different modes with different time delays. The selection of the matrices used for modal decomposition determines the model is either in phase-domain model or in modal-domain model. In phase-domain models a frequency-dependent matrix is used for modal decomposition however a fixed matrix at all frequencies is used in modal-domain models. This constant matrix can be eigenvector matrix at a specific frequency. As mentioned earlier, the phase-domain model is more accurate than the modal-domain model. The discrepancy between the two is more significant for cables.

All the models focus on the frequency domain model and do not consider the time domain realization effort and computation burden at modeling stage. Using real arithmetic instead of complex arithmetic in the implementation stage has been proposed in [8] and shown to decrease the number of floating point operations nearly by half. This work intends to improve the simulation speed by proposing a model to minimize the number of the states in the rational realization of the propagation function *H(s)*. The same method as [2] is used for characteristic admittance matrix *Y(s)*. For *H(s)* the poles are found based on [4] and for the residues, instead of a least square problem, a minimization problem is proposed which considers the rank of the residue matrices as a constraint. Manipulations are done to solve the non-convex optimization and the approach is verified for a transmission line case study system, for both the frequency response fitting and the time-domain simulation.

The rest of this paper is as following. First, the transmission line equations and the universal line model (ULM) are explained. Second, the proposed method is explained and the problem is reformulated. Third, the proposed method has been applied to a transmission line and the results are discussed. Time-domain results are also provided and compared to the PSCAD results for two different scenarios. The last part concludes the paper.

R. Mirzahosseini and R. Iravani are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, M5S 3G4 Canada (e-mail: ramin.mirzahosseini@mail.utoronto.ca).

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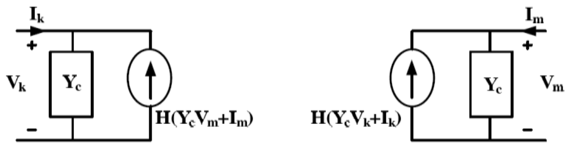

Fig. 1. Transmission Line phase Model

II. Transmission Line Phase-Domain Model

A. Formulation of Transmission Line Universal Model

A multiphase transmission line model is shown in Fig. 1. The equations describing the transmission line in the frequency domain are

$$\begin{aligned} I_k &= Y_c V_k - H(Y_c H_m + I_m), \\ I_m &= Y_c V_m - H(Y_c H_k + I_k). \end{aligned} \qquad (1)$$

Since the rational transfer functions are more appropriate for time domain simulations, two transfer function matrices Y_c and H in (1) should be approximated with rational transfer functions. The characteristic admittance matrix Y_c is smooth and can be fitted with a small number of poles. However, fitting the propagation function H, due to the presence of multiple delays, is more challenging. H is defined as

$$H = exp(-\sqrt{YZl}), \qquad (2)$$

where Y and Z are per unit length admittance and impedance matrices of the line, respectively. Eigenvalues of the matrix YZ in (2) are also eigenvalues of H by definition. Therefore eigenvectors of YZ can be used for modal decomposition of H as well. Thus the propagation matrix is decomposed into

$$\begin{aligned} H(\omega) &= T(\omega)\begin{bmatrix} e^{-\lambda_1(\omega)l} & 0 & \dots & 0 \\ 0 & e^{-\lambda_2(\omega)l} & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & e^{-\lambda_{n_m}(\omega)l} \end{bmatrix} T^{-1}(\omega) \\ &= \sum_{m=1}^{n_m} \Gamma_m(\omega) e^{-\lambda_m(\omega)l} \\ &= \sum_{m=1}^{n_m} \Gamma_m(\omega) e^{-\lambda'_m(\omega)l} e^{-j\omega\tau_m}, \end{aligned} \qquad (3)$$

where Γ_m is a square matrix which results from multiplying the m^{th} column of eigenvector matrix $T(\omega)$ by the m^{th} row of $T^{-1}(\omega)$, and τ_m is the appropriate delay for $e^{-\lambda m(\omega)l}$. This delay can be obtained by

$$\tau_m = \frac{l}{v_m(\Omega)}, \qquad (4)$$

where $v_m(\Omega)$ is the modal velocity defined separately for each mode and Ω is the highest frequency sample [4]. It should be noted that delay for minimizing the fitting error differs from (4) [5]. To obtain a rational approximation of H, approximations of each $\Gamma_m(\omega)e^{-\lambda' m(\omega)l}$ should be determined which are matrix transfer functions. This needs fitting of $\Gamma_m(\omega)$ and $e^{-\lambda' m(\omega)l}$ separately but in the universal model the approach

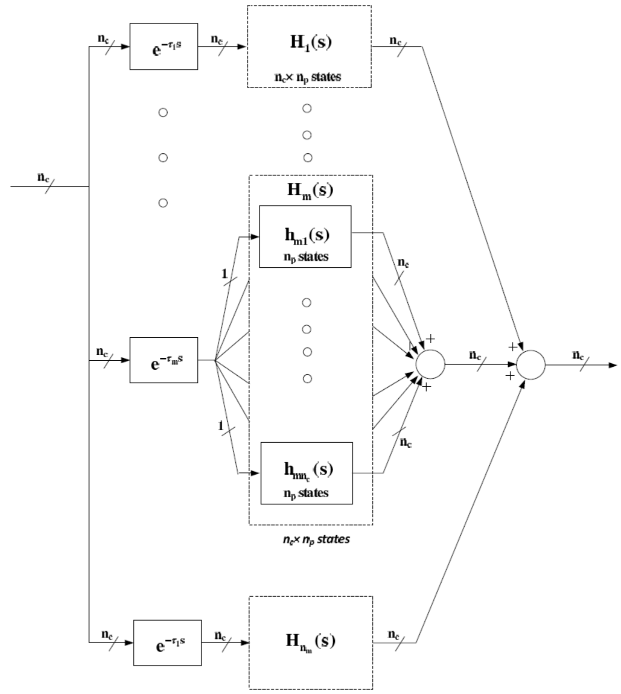

Fig. 2. Transmission Line phase model time domain realization

is different, i.e., first each $e^{-\lambda' m(\omega)l}$ is fitted which results in

$$e^{-\lambda'_m(\omega)l} \approx d_{m0} + \sum_{p=1}^{n_p} \frac{c_{mp}}{s - a_{mp}} \qquad (5)$$

Usually fitting problems are challenging since both poles and residues should be determined simultaneously. However, knowing these sets of poles from all modes, the propagation matrix transfer function fitting will turn into only finding residues as:

$$H_{ij}(j\omega_s) = \sum_{p=1}^{n_p}\left[\sum_{m=1}^{n_m} \frac{c_{mpij}}{j\omega_s - a_{mp}}\right] e^{-j\omega_s \tau_m}, \qquad (6)$$

where n_m and n_p are number of modes and the poles used in fitting of each mode respectively. It should be noted that since the eigenvectors matrix $T(\omega)$ and therefore $\Gamma(\omega)$ are smooth functions of frequency, then it is possible to compensate their effects by choosing appropriate residues. To determine the minimum error residues c_{mpij} the overdetermined matrix equation of the form (7)

$$Ax = B, \qquad (7)$$

where x is the vector of residues c_{mpij} and B is the vector of measured frequency responses is used.

B. Time Domain Implementation of ULM

The fitted propagation transfer function consists of multiple time delays and rational transfer function matrices as shown in (3). Since there are different time delays, each rational transfer function H_m is realized separately as follows. All entries of each H_m have the same set of poles so does each column of H_m called h_{mi}, as shown in (8).

$$H_m(s) = \begin{bmatrix} \sum_{p=1}^{n_p} \frac{c_{11pm}}{s-a_{pm}} & \cdots \\ \vdots & \sum_{p=1}^{n_p} \frac{c_{ijpm}}{s-a_{pm}} \end{bmatrix} \tag{8}$$
$$= \left[[h_{m1}(s)]_{n_c \times 1} \quad \dots \quad [h_{mn_c}(s)]_{n_c \times 1} \right].$$

Each h_{mi} is realized as a linear system with n_c outputs and one input. Considering the total number of columns of H_m to be n_c, the total number of poles used for realization of each H_m is $n_c \times n_p$. Therefore, the total number of poles for realization of *H(s)* is $\sum_{i=1}^{n_m} n_c \times n_{pi}$ where n_m is the number of modes, and usually is equal or less than the number of conductors n_c. In cases where there are multiple conductors, there are some modes which have the same time delays and considered as one mode. In addition, n_{pi} is the number of poles for i^{th} mode. It should be noted that without loss of generality it is assumed that the number of poles used in fitting of each mode are the same and referred to as n_p. Fig. 2 depicts the concept for realization of the propagation function *H(s)*.

III. Proposed Method

Each rational transfer function matrix in (8) can be written as

$$H_m(s) = \frac{[C_{1m}]_{n_c \times n_c}}{s - a_{1m}} + \dots + \frac{[C_{n_p m}]_{n_c \times n_c}}{s - a_{n_p m}}, \tag{9}$$

where each entry of C_{pm} is determined based on the least square problem (7). As mentioned previously and can be seen from Fig. 2, the order of the state-space realization of H_m is $n_c \times n_p$. All the entries in H_m share the same characteristic polynomial of order n_p. This means either the realization of H_m is not minimal or each pole is repeated n_c times where the latter scenario usually occurs. The number of times each pole a_p is repeated in realization of H_m is equal to the rank of the residue matrix C_{pm}. If C_{pm} is ideally rank one, then a_p will be a simple non-repeated pole. This indicates if all C_{pm} matrices are rank one, then the transfer function H_m can be realized with the n_p-order state-space model which reduces the calculation by factor of n_c. In this work it is intended to find a lower order state-space realization for H_ms. This order is desired to be n_p, since there are the same n_p poles in each entry of H_m. The order of the realized system for each H_m is

$$\deg(H_m) = \sum_{p=1}^{n_p} Rank(C_{pm}). \tag{10}$$

This goal can be obtained by minimizing the rank of C_{pm} matrices when solving (7). Solving the residue problem should be re-defined in terms of C_{pm} matrices as

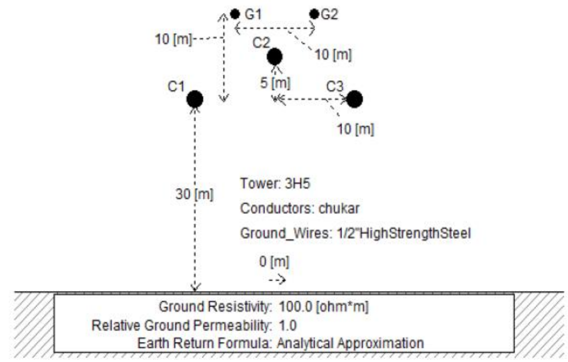

Fig. 3. Transmission line configuration

$$\min \sum_{m=1}^{n_m} \sum_{p=1}^{n_p} Rank(C_{pm}),$$
$$\text{subject to } \left\| H(j\omega_k) - H_d(j\omega_k) \right\| \le \varepsilon \quad k = 1, \dots, N_s,$$
$$C_{im} = \bar{C}_{jm} \quad if \quad a_i = \bar{a}_j, \tag{11}$$

where ε is the predefined maximum fitting error and N_s is the number of frequency sample points. This problem has been noticed in many fields including system identification [6] [7]. The rank minimization itself is a non-convex problem therefore difficult to solve. If there is a desired rank r for matrices C_{pm}, then the problem can be redefined as

$$\min \sum_{m=1}^{n_m} \sum_{p=1}^{n_p} \left\| H(j\omega_k) - H_d(j\omega_k) \right\|,$$
$$\text{subject to } Rank(C_{pm}) = r \quad p = 1..n_p,\ m = 1..n_m,$$
$$C_{im} = \bar{C}_{jm} \quad if \quad a_i = \bar{a}_j. \tag{12}$$

It is desired to have r=1 for the lowest order realization of order n_p for each H_m. Then, the above problem can be solved based on

$$Rank(C)_{n_c \times n_c} = 1 \Leftrightarrow C = u_{n_c \times 1} v_{n_c \times 1}^T, \tag{13}$$

which means instead of considering matrix C_{pm}, two vectors u_{pm} and v_{pm} can be considered. Therefore, the problem reduces to

$$\min \left\| H(j\omega_k) - H_d(j\omega_k) \right\|$$
$$\text{subject to } u_{im} = \bar{u}_{jm},\ v_{im} = \bar{v}_{jm}, \text{ if } a_i = \bar{a}_j, \tag{14}$$

where $H(j\omega)$ is a function of u_{pm}*s* and v_{pm}*s*. The formulation of the problem is as follows. All u_{pm}*s* and v_{pm}*s* are put in one column as variable *X* as

$$X = \begin{bmatrix} u_1^{11} \\ v_1^{11} \\ \vdots \\ u_i^{m_p} \\ v_i^{m_p} \\ \vdots \\ u_{n_c}^{n_m n_p} \\ v_{n_c}^{n_m n_p} \end{bmatrix}_{2n_c n_m n_p \times 1}, \quad u^{mp} = \begin{bmatrix} u_1^{mp} \\ u_2^{mp} \\ \vdots \\ u_{n_c}^{mp} \end{bmatrix}, \quad v^{mp} = \begin{bmatrix} v_1^{mp} \\ v_2^{mp} \\ \vdots \\ v_{n_c}^{mp} \end{bmatrix}, \tag{15}$$

and $F(X)$ is

$$F(x) = \left[H(j\omega_k) - H_d(j\omega_k) \right] = \begin{bmatrix} f_{11}^1(X) \\ f_{12}^1(X) \\ \vdots \\ f_{ij}^k(X) \\ \vdots \\ f_{n_c n_c}^{N_s}(X) \end{bmatrix}, \tag{16}$$

where $H_d(j\omega)$ is the data to be fitted and

$$\begin{aligned} f_{ij}^k(X) &= \Big(\sum_{m=1}^{n_m} \sum_{p=1}^{n_p} e^{-j\tau_m \omega_k} \frac{C_{ij}^{mp}}{j\omega_k - a_p} - H_{ij}^d(j\omega_k)\Big) \\ &= \Big(\sum_{m=1}^{n_m} \sum_{p=1}^{n_p} e^{-j\tau_m \omega_k} \frac{u_i^{mp} v_j^{mp}}{j\omega_k - a_p} - H_{ij}^d(j\omega_k)\Big). \end{aligned} \tag{17}$$

Finally the minimization can be expressed as

$$\min\ g(X)$$
$$\text{subject to } u_{im} = \bar{u}_{jm},\ v_{im} = \bar{v}_{jm},\ \text{if } a_i = \bar{a}_j, \tag{18}$$

where $g(X) = \|F(X)\|_2$. It should be noted that this problem is different from linear least square problem in the universal line model. This problem is a non-linear minimization problem and can be solved using gradient method. To do this, $\nabla g(X)$ should be calculated, i.e.,

$$\nabla g(X) = \frac{J_F(X)^T F(X)}{\|F(X)\|_2} = \frac{\sum_{i,j,k} (f_{ij}^k(X) \nabla f_{ij}^k(X))}{\|F(X)\|_2}. \tag{19}$$

The approach was tested on a three phase transmission line propagation matrix for the line configuration shown in Fig. 3. The results are obtained after a large number of iterations and the convergence rate is very slow. The reason is that the initial X is considered to have entries all equal to 0.1 and this results in two problems. First, due to the $\nabla f_{ij}^k(X)$ structure, if all entries of initial vector X are the same, they remain the same during all iterations. The second problem is that the final residues are in a very wide range since the poles are from 1Hz to 1MHz. Therefore, starting from the initial conditions which have the same entry range makes the problem ill conditioned. This can be resolved by normalizing the residues by their corresponding poles as

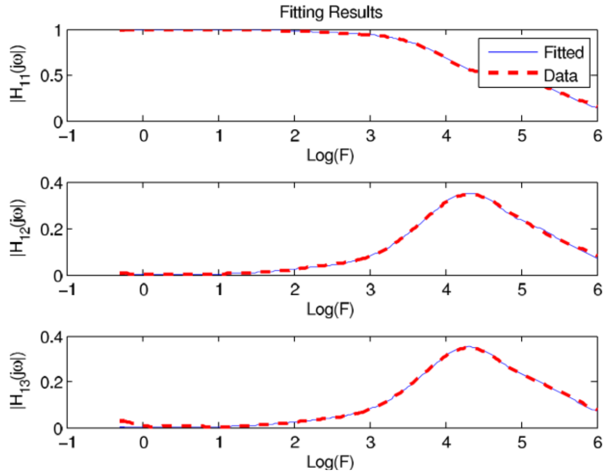

Fig. 4. Fitting result for rank one propagation transfer function matrix fitting with random equal initial condition for normalized residue C'_{pm} and its vectors u'_i and v'_i where all elements of the initial vectors were considered to be in the range of (0,1)

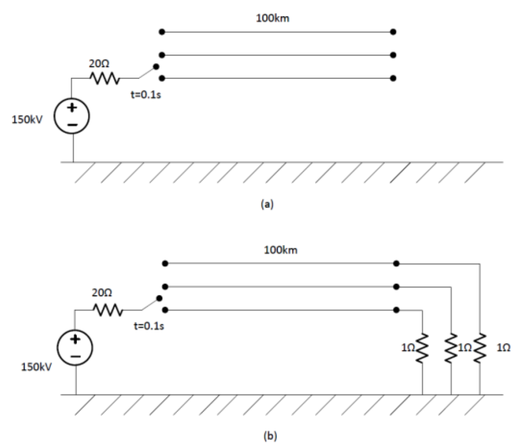

Fig. 5. Two different scenarios for comparing step response, (a): open circuit, (b): three phase short circuit

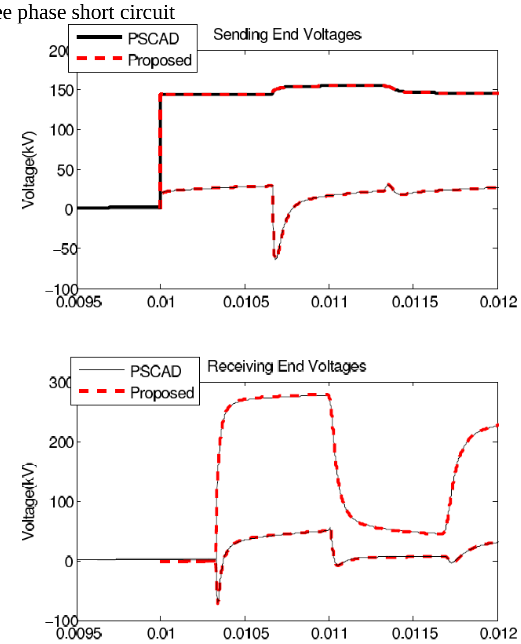

Fig. 6. Time-domain simulation result of the proposed phase model for transmission line open circuit step response voltages at both sending and receiving voltages

$$H_m(s) = \frac{[C'_{1m}]_{n_c \times n_c}}{\dfrac{s}{a_{1m}} - 1} + .. + \frac{[C'_{n_p m}]_{n_c \times n_c}}{\dfrac{s}{a_{n_p m}} - 1}, \qquad (20)$$

where $C'=C_{pm}/a_{pm}$. Using this approach, the minimization converges much faster and the initial X is considered to be random to solve the first problem. The results of the minimization are shown in Fig. 4 which favorably match the data with an acceptable accuracy.

IV. Time Domain Simulation Results

To verify functionality of the method the transmission line in Fig. 3 is simulated with the proposed method based on a MATLAB code and compared to the PSCAD results. It should be noted that the same number of poles was used for approximation of the propagation function matrix in both cases. However, one third of the number of states is used in the proposed method which means reducing the computation by a factor of three, i.e., the same as the number of conductors.

Two different scenarios are considered as shown in Fig. 5, i.e., open circuit and short circuit tests of the transmission line at the receiving end. The sending and receiving phase voltages are shown in Fig. 6 and Fig. 7 which accurately compare with the PSCAD results. Results for the short circuit test are depicted in Fig. 8 and Fig. 9 and also show good match with the PSCAD results.

V. Conclusion

This paper presents an improved method for modeling of the transmission line propagation function. In this approach the characteristic admittance matrix is modeled in the same way in transmission line universal model, i.e., using vector fitting for transfer function matrix approximation. For the propagation function approximation, instead of using a simple overdetermined minimization which results in full-rank residue matrices, rank one minimization is used to enable minimal realizations with a smaller number of states for time domain integration of propagation matrix transfer function. This minimization problem was manipulated to obtain and demonstrate acceptable results and fast convergence by proper variable normalization. A case study system was considered and the results for both frequency-domain approximation and time-domain were presented. The results closely agree with those obtained from the PSCAD-based simulation results. Although the model has the advantage of less computation burden during simulation at the cost of longer fitting time.

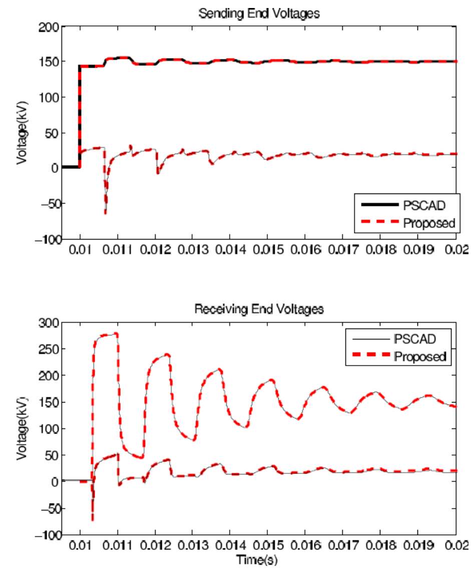

Fig. 7. Long snap shot of time domain simulation result of the proposed phase model for transmission line open circuit simulation results

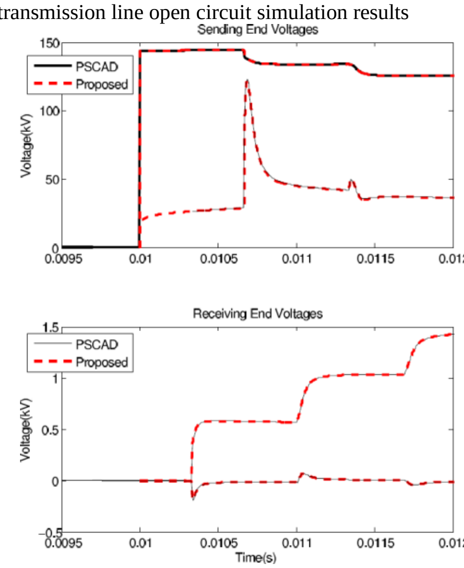

Fig. 8. Time-domain simulation result of the proposed phase model for transmission line three phase short circuit step response voltages at both sending and receiving voltages

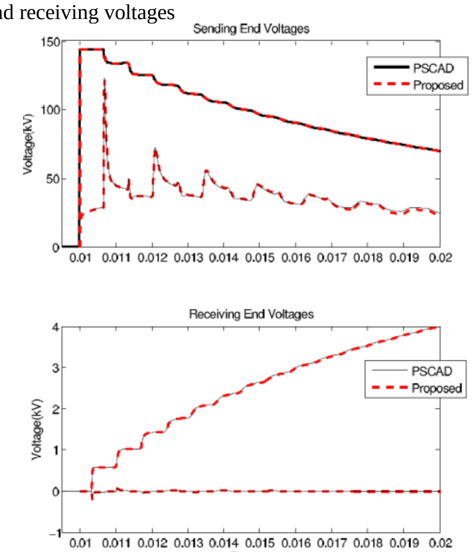

Fig. 9. Long snap shot of time domain simulation result of the proposed phase model for transmission line short circuit simulation results

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