

A Study on Interpolation and Weighting Function for Numerical Fourier Transform

Xi Shi, Akihiro Ametani, Aniruddha M. Gole

Abstract—In order to mitigate the Gibbs oscillation, a very simple and effective linear mid-point interpolation method is proposed. The relationship between proposed linear mid-point interpolation in time domain and window function in numerical inverse Fourier transform is also investigated in this paper. It is proved that the linear mid-point interpolation in time domain is equivalent to the cosine window function defined as $G_{cos}(\omega) = \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_{max}}\right)$, where ω_{max} is the maximum angular frequency used in the transform. Furthermore, the cosine window function and sinc window function (also known as sigma-factor) show the similar characteristic. A weighting order n , which is originally defined as the power to which the window function is raised, can also be applied to the interpolation method when n is an integer. The n^{th} -time interpolation is equivalent to applying the window function $[G_{cos}(\omega)]^n$ in frequency domain.

Keywords: numerical Fourier transform, weighting function, interpolation, Gibbs oscillation.

I. INTRODUCTION

Gibbs phenomenon describes the large overshoot and oscillations of the Fourier series at the jump discontinuity, which was first discovered by Henry Wilbraham in 1848 [1]. It was widely believed the oscillation was due to the flaws of the device when re-synthesize the Fourier series. In 1899, Gibbs published a description of the overshoot at the point of discontinuity in [2]. In 1906, a mathematical analysis was given by Maxime Bocher [3]. It's found out that the overshoot does not die out with the increase of the Fourier terms, but the span can be shortened, while the height of the overshoot remains the same.

In order to alleviate the oscillation, various methods were used, including Fejér summation [4] or Riesz summation [5], sigma-approximation [6], discrete wavelet transform with Haar basis functions [7], etc. In this paper, a very simple, easy-to-use, after computation method is proposed to effectively mitigate Gibbs oscillation.

In Section II, the cause of the Gibbs phenomenon and the sinc window approach have been reviewed. In Section III, the linear mid-point interpolation method is introduced; the proof of the equivalence of the linear mid-point interpolation method and cosine window is given; the effect on Gibbs

oscillation suppression is compared with sinc window method. In Section IV, the weighting order is applied to linear mid-point interpolation method. In Section V, the proposed approach is used in the switching surge simulation. The findings are concluded and summarized in Section VI.

II. GIBBS PHENOMENON AND SINC WINDOW

To get the time domain response, inverse Fourier transform can be applied to the frequency domain functions [8], [9],

$$f(t) = \frac{1}{\pi} \int_0^{\infty} F(\omega) \exp(j\omega t) d\omega \quad (1)$$

where $F(j\omega)$ is the frequency domain function to be evaluated. When (1) is evaluated numerically, the integration range $[0, \infty]$ must be substituted by a finite range $[0, \omega_{max}]$ with sufficiently large ω_{max} as in (2).

$$f(t) = \frac{1}{\pi} \int_0^{\omega_{max}} F(\omega) \exp(j\omega t) d\omega \quad (2)$$

When the frequency domain function is numerically evaluated, the integration range is abruptly chopped at a certain frequency ω_{max} . As a result, the original signal $F(\omega)$ can be seen to be multiplied by a “brick-wall” window function $G(\omega)$. From the signal processing view, the window function $G(\omega)$ can be seen as an ideal low-pass/high-cut filter. The window function $G(\omega)$ is defined in (3) and is illustrated in Fig. 1.

$$G(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_{max} \\ 0 & \text{else} \end{cases} \quad (3)$$

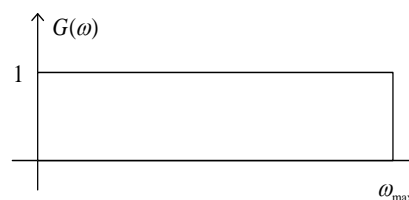


Fig. 1. Window function $G(\omega)$

This ideal rectangular window function $G(\omega)$ in the

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frequency domain corresponds to the sinc function $g(t)$ (also called the "sampling function") in the time domain. $g(t)$ is defined in (4) and is illustrated in Fig. 2.

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}(G(\omega)) \\ &= (\omega_{max}/\pi) \frac{\sin(\omega_{max}t)}{\omega_{max}t} \\ &= (\omega_{max}/\pi) \text{sinc}(\omega_{max}t) \end{aligned} \quad (4)$$

where \mathcal{F}^{-1} denotes inverse Fourier transform. With $f_{max} = 10^7$ Hz ($\omega_{max} = 2\pi \times 10^7$ rad/s), the sinc function in the time domain is as shown in Fig. 2.

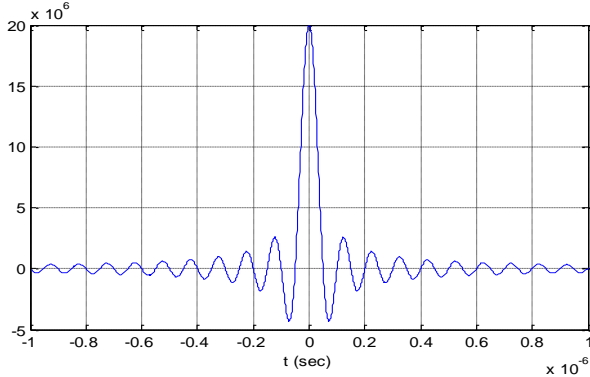


Fig. 2. Sinc function in time domain

The product in frequency domain $F(\omega)G(\omega)$ is equal to convolution in the time domain $f(t) * g(t)$. The oscillations of the sinc function $g(t)$ causes the ripples in the output result. Imagine $f(t)$ is a step function which jumps from 0 to 1 at $t = 0$ sec, then the convolution of $f(t) * g(t)$ results in the overshoot whose magnitude is thus exactly the integral of the (left) tail.

Because the ripples are caused by the oscillations of the sinc function $g(t)$, and the frequency of the oscillations of sinc function is determined by the upper limit of the frequency range during integration, which can be seen as equal to f_{max} as in Fig. 2. If the output time step is equal to or larger than $1/f_{max}$, then the ripples may not be observed. The same case of a step function is calculated with different time steps, and the results are shown in Fig. 3. The upper limit of the integration $f_{max} = 10^7$ Hz, the timestep Δt for the blue curve is smaller than $1/f_{max}$ while the timestep for the red curve is equal to $1/f_{max}$. It's obvious that if the timestep $\Delta t \geq 1/f_{max}$, there may be unnecessary to suppress Gibbs oscillation.

It's also noticeable from Fig. 3 that the red curve with a larger timestep also loses some accuracy. Therefore, in all transmission line simulations in the following chapters, the timestep used is no larger than $\frac{1}{10}$ of one cycle of the oscillation, e.g. $\Delta t \leq \left(\frac{1}{10}\right)(1/f_{max})$. Then the Gibbs oscillation suppression approach is a necessity in the simulation if the discontinuity is involved.

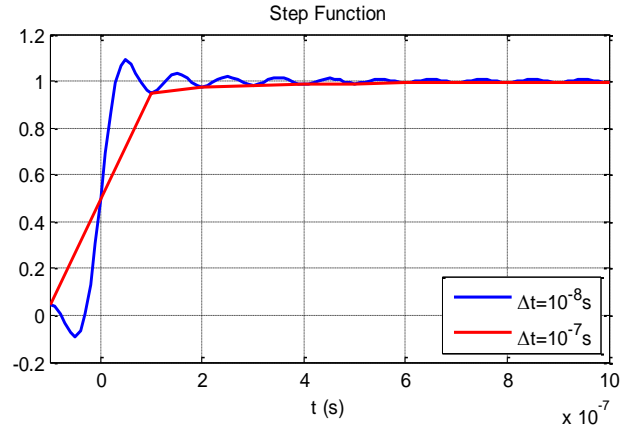


Fig. 3. Calculated results with different time steps

The well-known method to eliminate Gibbs phenomenon is that $F(\omega)$ is weighted with a less abrupt window function $G_{\text{sinc}}(\omega)$ derived by S. J. Day, N. Mullineux and J. R. Reed [6] with the following equation. (Note that this sinc window $G_{\text{sinc}}(\omega)$ is in frequency domain, while the sinc window function $g(t)$ in (4) is in time domain.)

$$G_{\text{sinc}}(\omega) = \begin{cases} \frac{\sin\left(\pi \frac{\omega}{\omega_{max}}\right)}{\left(\pi \frac{\omega}{\omega_{max}}\right)}, & 0 \leq \omega \leq \omega_{max} \\ 0, & \text{else} \end{cases} \quad (5)$$

The curve of this $G_{\text{sinc}}(\omega)$ window function is shown in Fig. 4. It's can be seen that the value remains 1 for the most part of the frequency range, but instead of drops to 0 abruptly at a frequency ω_{max} , it drops to 0 gradually and smoothly as the frequency approaches ω_{max} .

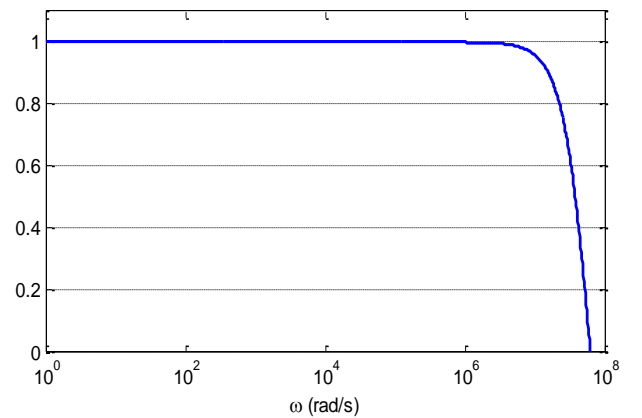


Fig. 4. $G_{\text{sinc}}(\omega)$ window function with $f_{max} = 10^7$ Hz

We can see from Fig. 5, after applying sinc window (which is also called σ -approximation [10], [11]), the oscillation is greatly reduced compared to the original result.

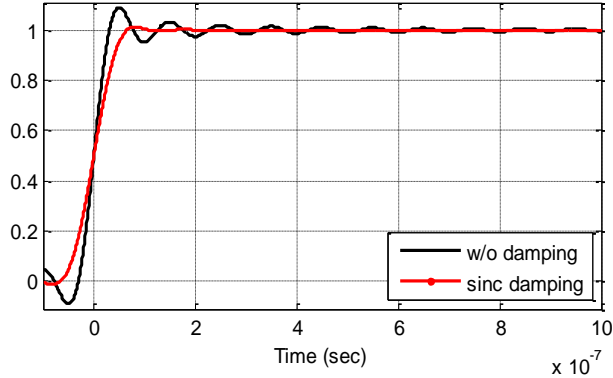


Fig. 5. Oscillations damped with sinc method

This method requires the calculation of $G_{\text{sinc}}(\omega)$ at each frequency point and also the multiplication of $G_{\text{sinc}}(\omega)$ with $F(\omega)$ at each frequency point. In the next section, a new approach to mitigate the Gibbs oscillation is proposed which effectively reduces the oscillation and is computationally much more efficient. We have labeled it “linear mid-point interpolation”.

III. PROPOSED LINEAR MID-POINT INTERPOLATION METHOD

To reduce the oscillation, a new and more computationally efficient method is proposed in this paper. It performs a linear mid-point interpolation after the original calculated result (the one with Gibbs oscillation) is obtained. With the sinc window, $F(\omega)$ must be multiplied by $G_{\text{sinc}}(\omega)$ at each frequency point and then the product to be integrated as in inverse Fourier transform. As an alternative, the proposed “linear mid-point interpolation” method can be utilized after the result curve is obtained from (2) and this can be done only when researchers feel necessary after observing the existence of Gibbs oscillation. This method is illustrated in Fig. 6. Let’s assume that the samples of the original result with a time-step Δt , at $t = t_0, t_0 + \Delta t, t_0 + 2\Delta t, \dots$ are $f(t_0), f(t_1), f(t_2), \dots, f(t_n)$, respectively. These are used to calculate the new values $\bar{f}(t_1), \bar{f}(t_2), \dots, \bar{f}(t_n)$, $t_k = t_0 + k\Delta t, k = 1, 2, \dots, n$ as in (6).

$$\bar{f}(t_i) = \frac{1}{2} \{f(t_{i-K}) + f(t_{i+K})\}, K \leq i \leq n - K \quad (6)$$

where $K = \frac{T_0}{\Delta t}$, Δt is the output time step, the time interval T_0 is defined as $T_0 = \frac{1}{4f_{\text{max}}}$.

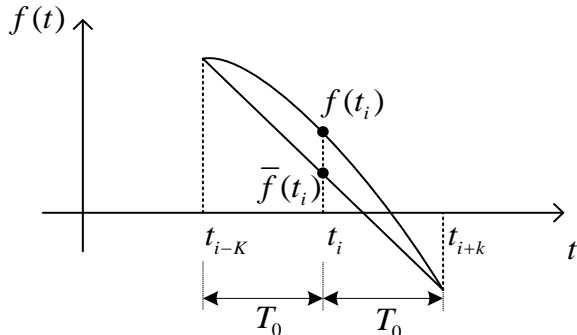


Fig. 6. Illustration of the linear mid-point interpolation

Equation (6) can be rewritten, from a point of view of convolution, in the following equation

$$\bar{f}(t) = \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau \quad (7)$$

where $g(t) = \frac{1}{2}(\delta(t + T_0) + \delta(t - T_0))$ and $\delta(t)$ is the Dirac delta function.

It means $\bar{f}(t)$ is the convolution of $f(t)$ and $g(t)$; $g(t)$ is one half of the sum of two $\delta(t)$ functions with a time offset of $+T_0$ and $-T_0$ respectively.

The convolution of $f(t)$ with $\delta(t)$ function in the time domain is equal to that $F(\omega)$ times 1 in the frequency domain, and the time offset of T_0 in time domain is equal to that $F(\omega)$ times $e^{-j\omega T_0}$. Therefore, the time domain equation (7) is equivalent to the frequency domain equation (8).

$$\begin{aligned} \bar{F}(\omega) &= \frac{1}{2}(F(\omega)e^{-j\omega T_0} + F(\omega)e^{j\omega T_0}) \\ &= F(\omega) \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_{\text{max}}}\right) \end{aligned} \quad (8)$$

Similar to sinc window method, $\bar{F}(\omega)$ can be seen as $F(\omega)$ weighted by a cosine window function $G_{\text{cos}}(\omega) = \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_{\text{max}}}\right)$. The cosine window function $G_{\text{cos}}(\omega)$ is shown in Fig. 7 as the blue curve, and the sinc window function $G_{\text{sinc}}(\omega)$ is also in red curve for comparison. Like the sinc window function, the cosine window function remains 1 for the most part of the frequency range and drops to 0 gradually as the frequency approaches ω_{max} . This explains the mechanism of the interpolation method in suppression of oscillation.

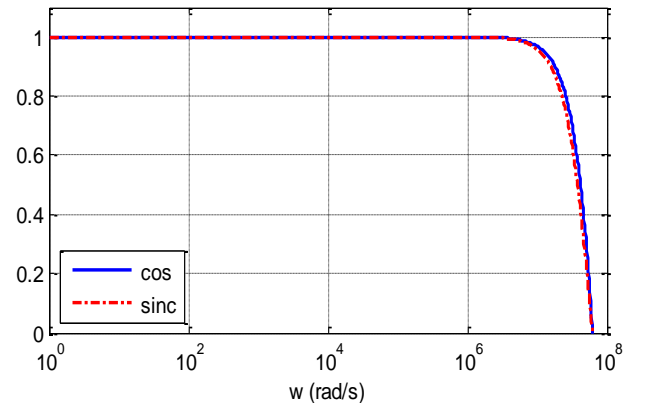


Fig. 7. $G_{\text{cos}}(\omega)$ and $G_{\text{sinc}}(\omega)$ window function with $f_{\text{max}} = 10^7$ Hz

It is interesting to mention that the linear mid-point interpolation method is also used in EMT-type programs to suppress the numerical oscillations (chatter) caused by discontinuities (switching, the action of power electronics, etc.), which is usually named as half step interpolation. The idea of using half step interpolation to suppress oscillations probably comes from the intuition that the linear mid-point (the average value) of the line connecting the adjacent two

points lies on the correct solution, since the two points with numerical oscillations evenly distributed on the two sides of real solution [12]-[16]. However, the above analysis explains the plausibility of the half step interpolation method from a different viewpoint. The half step interpolation is actually a low pass filter (or multiply cosine window function in the frequency domain), which can filter the high frequency oscillations.

This cosine window method can also be implemented in the frequency domain. The result obtained from the cosine window function method is compared with the one obtained from the linear mid-point interpolation method, which is shown in Fig. 8. It is clear that the cosine window function method and the linear mid-point interpolation method obtain identical results, which proves the linear mid-point interpolation method is not only equivalent to cosine window function method in theory, but also the two are equivalent in practice.

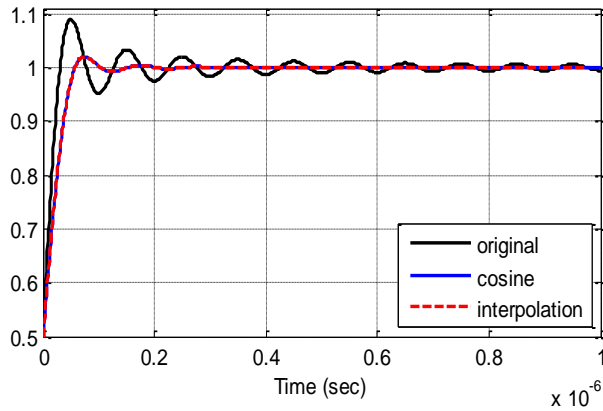


Fig. 8. Cosine window function and linear mid-point interpolation comparison

In Fig. 9, the linear mid-point interpolation method is also compared with sinc method. Both methods have reduced the Gibbs oscillation dramatically and the difference between these two methods is marginal.

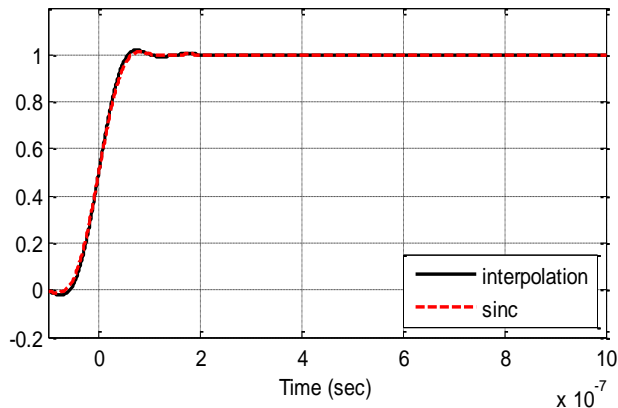


Fig. 9. The linear mid-point interpolation and sinc method comparison

IV. WEIGHTING ORDER

From Fig. 9, we can see that the oscillation is greatly reduced but not eliminated. To obtain better results, we can apply a weighting order n to sinc window function (or cosine

window function), then the new window function $G_{\text{sinc}}^n(\omega) = [G_{\text{sinc}}(\omega)]^n$ (or $G_{\text{cos}}^n(\omega) = [G_{\text{cos}}(\omega)]^n$), where n can be any real number larger than zero. As weighting order n gets larger, the Gibbs oscillation gets more suppressed. When the weighting order n is an integer number, the cosine window function to the power of n $G_{\text{cos}}^n(\omega)$ can also be implemented with the linear mid-point interpolation method, and n will be the rounds of the linear mid-point interpolation applied in the time domain. Take $n=2$ as an example to prove the statement. The value after second round of linear mid-point interpolation $\bar{\bar{f}}(t_i)$ is as shown in (9).

$$\bar{\bar{f}}(t_i) = \frac{1}{2} \{ \bar{f}(t_{i-K}) + \bar{f}(t_{i+K}) \} \quad (9)$$

$\bar{f}(t_i)$ in (9) are obtained from the first round of linear mid-point interpolation, and as shown in (10) and (11).

$$\bar{f}(t_{i-K}) = \frac{1}{2} \{ f(t_{i-2K}) + f(t_i) \} \quad (10)$$

$$\bar{f}(t_{i+K}) = \frac{1}{2} \{ f(t_i) + f(t_{i+2K}) \} \quad (11)$$

Substitute (10) and (11) into (9),

$$\bar{\bar{f}}(t_i) = \frac{1}{4} f(t_{i-2K}) + \frac{1}{2} f(t_i) + \frac{1}{4} f(t_{i+2K}) \quad (12)$$

Similarly, (12) can be written, from the viewpoint of convolution, as in (13),

$$\bar{\bar{f}}(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \quad (13)$$

where $g(t) = \frac{1}{4} (\delta(t+2T_0) + \delta(t-2T_0)) + \frac{1}{2} \delta(t)$.

Transform the time domain equation (13) into a frequency domain equation,

$$\begin{aligned} \bar{\bar{F}}(\omega) &= \frac{1}{4} (F(\omega) e^{-j\omega 2T_0} + F(\omega) e^{j\omega 2T_0}) + \frac{1}{2} F(\omega) \\ &= \frac{1}{2} F(\omega) \left[\cos\left(2 \times \frac{\pi}{2} \frac{\omega}{\omega_{\max}}\right) + 1 \right] \\ &= \frac{1}{2} F(\omega) \left[2 \times \cos^2\left(\frac{\pi}{2} \frac{\omega}{\omega_{\max}}\right) - 1 + 1 \right] \\ &= F(\omega) \cos^2\left(\frac{\pi}{2} \frac{\omega}{\omega_{\max}}\right) \\ &= F(\omega) G_{\text{cos}}^2(\omega) \end{aligned} \quad (14)$$

From the above equation, the second round of linear mid-point interpolation is clear to be equivalent to multiply $G_{\text{cos}}^2(\omega)$ in the frequency domain. In the similar way, n can be generalized to other integers, where the n -th round of linear mid-point interpolation method is equivalent to multiply the n -th power of $G_{\text{cos}}(\omega)$ in frequency domain. The effect of different weighting orders on step function is shown in Fig. 10 for the linear mid-point interpolation method. It's clear that the overshoot gets smaller as n gets larger.

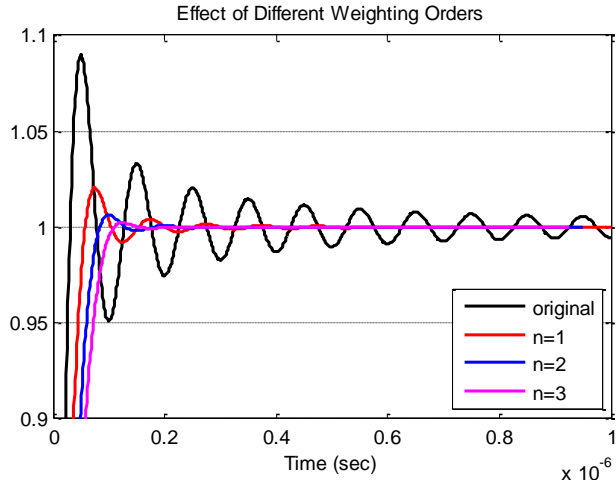


Fig. 10. Weighting order effects for the linear mid-point interpolation method

V. CALCULATION EXAMPLE

The frequency domain simulation is widely used to verify the frequency-dependent transmission line models[17]-[20]. Fig. 11 illustrates model circuits for transient studies in this paper. The switching surges on a single conductor above the imperfectly conducting earth with resistivities ρ_e is simulated. The conductor height $h = 15$ m, radius $r = 1$ cm, length $x = 1$ km, and the conductor resistivity is $\rho_c = 2 \times 10^{-8} \Omega\text{m}$. Switch S is closed at $t = 0$ s and the switching surge voltage V_r at the open-circuited receiving end are calculated.

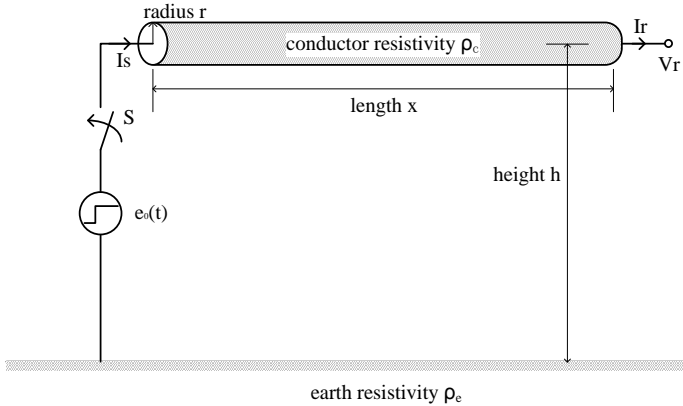


Fig. 11. Switching surge on a single conductor

The parameters of the conductor in this paper are calculated as follows [21]

$$Z = j\omega(\mu_0/2\pi)\ln(S/r) \quad (15)$$

$$Y = j\omega C_0 = j\omega 2\pi\epsilon_0/\ln(2h/r) \quad (16)$$

where $S^2 = 4(h + h_e)^2 + r^2$, $h_e^2 = \rho_e/(j\omega\mu_0)$, Z is the per-unit-length impedance and Y is the per-unit-length admittance.

The relationship of each variable is given as below [21],

$$\begin{pmatrix} V_r \\ I_r \end{pmatrix} = \begin{bmatrix} \cosh(\gamma x) & -Z_0 \sinh(\gamma x) \\ -Y_0 \sinh(\gamma x) & \cosh(\gamma x) \end{bmatrix} \begin{pmatrix} V_s \\ I_s \end{pmatrix} \quad (17)$$

where γ is the propagation constant, Z_0 is the characteristic impedance and Y_0 is the characteristic admittance, which can

be calculated as in (18).

$$\begin{cases} \gamma = \sqrt{ZY} = \alpha + j\beta \\ Z_0 = \sqrt{Z/Y} \\ Y_0 = 1/Z_0 \end{cases} \quad (18)$$

Source voltage V_s at the sending end is set to be 1 p.u., and the receiving end is open, i.e. the receiving end current I_r is 0 and then we can have receiving end voltage V_r as below,

$$V_r = \frac{1}{\cosh(\gamma x)} V_s = \frac{1}{\cosh(\gamma x)} \quad (19)$$

Applying inverse Fourier transform to (19), we can get the time domain response of the switching surge at the open end. The original result with Gibbs oscillation is shown in Fig. 12 as black curve. Due to the travelling wave reflections, we can see the receiving end voltage jumps between 0 and 2 p.u. before it reaches steady state. At each discontinuity of the black curve, Gibbs oscillations are observed.

After the original black curve is obtained, the linear mid-point interpolation method is applied to the black curve. The first round interpolated result is shown in Fig. 12 as red curve. Then the mid-point interpolation method is applied to the red curve again to obtain the second round interpolated result, which is shown in Fig. 12 as blue curve.

It's can be seen from Fig. 12 that the three curves agree well at the smooth parts, while at discontinuous points, the Gibbs oscillation is greatly reduced after the first round of interpolation ($n = 1$) and is even more suppressed after the second round of interpolation ($n = 2$).

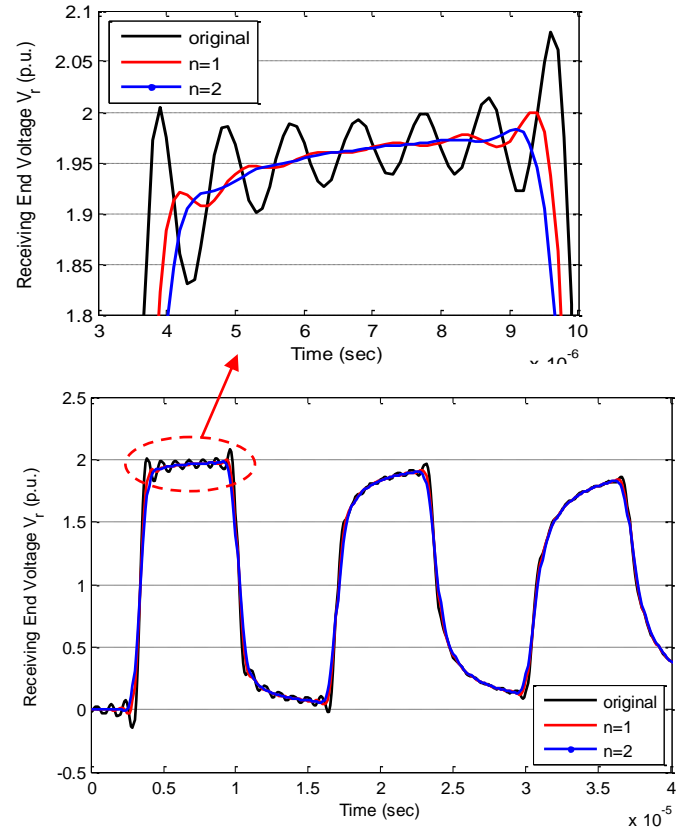


Fig. 12. Simulation result of switching surge

VI. CONCLUSIONS

A very simple and effective approach to suppress the Gibbs oscillation is discussed in detail in this paper. The linear mid-point interpolation method is proved to be equivalent to the multiplication of cosine window function in the frequency domain. The comparison of cosine window with sinc window further shows the plausibility of the cosine window. Because of the equivalency, the plausibility of the interpolation method is naturally demonstrated. Furthermore, a higher weighting order n in the frequency domain is proved to be equivalent to multiple interpolations (when n is an integer), which is shown to further suppress Gibbs oscillations as n becomes larger. The method is then applied to the simulation of switching surge on a single conductor, and a satisfactory result is obtained through the proposed linear mid-point interpolation method.

VII. ACKNOWLEDGEMENT

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Dear Sir/Madam,

We wish to submit the paper "A Study on Interpolation and Weighting Function for Numerical Fourier Transform" for consideration by IPST 2021.

Due to its ability to accurately include the frequency-dependent characteristics, the frequency domain simulation is widely used to verify the time domain models. However, the intrinsic Gibbs oscillation occurs at discontinuity when inverse Fourier transform is numerically evaluated. The current solutions of this issue are more complicated than the method proposed in our paper, which is simple, effective and can be done after the original result is obtained. Although the proposed method is originally used for over-head transmission line simulations, it also can be applied to any simulations where inverse Fourier transform is involved.

Should you have any questions concerning this manuscript, please feel free to contact me at shix34@myumanitoba.ca.

Thank you for your consideration of this paper.

Sincerely,

Xi Shi, M.Sc.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Aniruddha Gole

雨谷 昭弘 (Ametani Akihiro)