

An Efficient Analytical Based Technique to Numerical Calculation of Extended Earth Return Impedance and Admittance of Overhead Lines

Z Liu, H.M.J. De Silva

***Abstract*—This paper describes an efficient numerical integration technique for the extended overhead line earth return impedance and admittance formula. In-appropriate handling of the infinite integral in the formula can lead to erroneous calculated earth-return parameters at extreme frequencies, and a significant increase in computational effort. In this paper, an efficient technique based on an analytical approach is introduced to enforce proper discretization of the extended transmission line equations. Firstly, the line equations are analyzed, and a common cause of improper discretization is identified. Then the equations are further broken down numerically, based on which a procedure to suitably select a discretization interval and step is developed. Validation was conducted by calculating the earth return parameters of overhead-line cases using the proposed method and equal distance discretization. The proposed technique shows accurate results while using fewer discretization points than equal distance discretization.**

***Keywords*: Electromagnetic transient modelling, Earth return impedance and admittance, Numerical integration, Overhead lines**

I. Introduction

ELECTROMAGNETIC transient study of power system requires transmission line's earth return parameters to be accurately evaluated in a wide range of frequencies [1-4]. The frequency of interest can vary from 0 Hz for DC operation [5], to serval MHz for lightning transient, and Gas insulated substation studies [6-7].

Carson/Pollaczek's earth-return impedance and space admittance (classical transmission line (TL) approach) are widely used in EMT programs [8],[9]. The recent extended TL approach with accurate earth-return impedance and admittance [10],[11] improves the accuracy for a wide frequency range and more importantly enhances the stability of the time domain simulation [7].

Numerous amounts of previous studies on proper implementation of the classic TL formula has been conducted, but not for the extended TL formula. Although they are technically different, it is still possible to develop similar implementation methods by correlating with pervious works. This is due to the similarity of complexity in the formula themselves (i.e. integrand components). Previous studies have developed serval different approaches to the classic TL formulas. These includes generalized discretization variable [12], solving the infinite integral via series expansion [1], Taku Noda's integral transformation techniques [13], asymmetric extraction method [2], or custom partitioning of integrand and applying various suitable integration method to each partitions [14]. However, most of the time the evaluation of integration variables such as truncation length, and discretization step size is not straightforward. This leads to the use of recursive algorithms which enforces convergence by recalculating using different integration parameters repeatedly. To avoid recursive methods, past researchers have developed analytical based approach for Pollaczek's underground cable earth impedance [15], however, no similar methods has yet been conducted for the extended TL formula for overhead line.

This paper proposes a systematic procedure derived analytically to compute the earth return parameters extended TL formulas accurately, numerically efficiently, and without recursive steps.

II. Review of Overhead Line equations

A. The approximate overhead line earth return formula

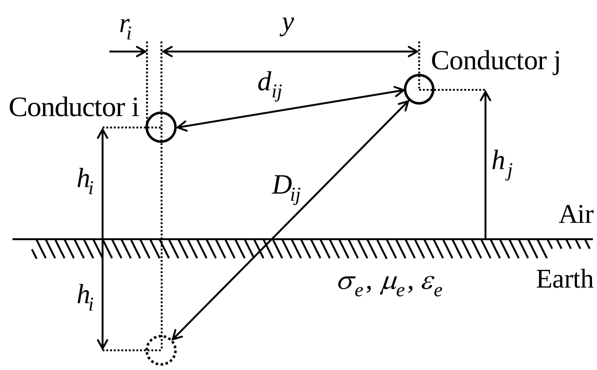

Figure 1. Transmission line configuration with overhead lines and cables

In this paper, Z, Y, R, L, G, C, f, ω represents per unit length impedance, admittance, resistance, inductance, conductance, capacitance, frequency, angular frequency respectively.

Figure 1 shows a typical two conductors overhead transmission line system. Where, $\sigma_e, \mu_e, \epsilon_e, y, h$ represent earth resistivity, permeability, permittivity, line separation, and

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height above ground respectively. Traditionally, the overhead line earth return value is solved using formulas developed by J. R. Carson in 1926 [8]. One difficulty arises when adopting the Carson formula is the high computation effort required to evaluate its infinite integral. Hence, later an approximate formula to represent Carson's integral is derived by Gary [16], [17]. The closed form approximate was further worked on and refined by many engineers such as Alvarado [18], Noda [19]. To accommodate for displacement this paper will use the approximate modified Carson's formula given in (1) [20].

$$Z = j\omega(\mu_0/2\pi)\ln(S_{ij}/d_{ij}) \quad \left[\frac{\Omega}{m}\right] \tag{1}$$

where,

$$\omega = 2\pi f \tag{2}$$

$$S_{ij} = (h_i + h_j + 2h_e)^2 + y^2 \tag{3}$$

$$h_e = \frac{1}{\sqrt{j\omega\mu_0(\sigma_e + j\omega\epsilon_0(\epsilon_r - 1))}} \tag{4}$$

$$d_{ij} = (h_i - h_j)^2 + y^2 \tag{5}$$

This approximate formula has the benefit of taking in consideration of displacement current which most other closed form approximate does not consider. The closed form approximate formula for earth return admittance is calculated using the well-known formula in (6). The closed form approximate was further improved by other researchers such as D'Amore [21]

$$Y = \frac{j\omega\mu_0}{2\pi}\ln\left(\frac{S_{ij}}{d_{ij}}\right) \quad \left[\frac{S}{m}\right] \tag{6}$$

B.The extended overhead line formula

The extended earth-return impedance of a multi-phase overhead line is given as (7) [10],[22]. The extended earth-return admittance by Nakagawa is given in (8) which is the same as that derived by Wise [11],[23].

$$Z_{ij} = j\omega\left(\frac{\mu_0}{2\pi}\right)[P_0 + (Q - jR)] \tag{7}$$

$$P_{ij} = \frac{P_0 + M + jN}{2\pi\epsilon_0} \tag{8}$$

where,

$$P_0 = \ln\left(\frac{D_{ij}}{d_{ij}}\right) \tag{9}$$

$$Q - jR = 2\int_0^\infty F_1(s)\,ds \tag{10}$$

$$F_1(s) = \frac{e^{-(h_i+h_j)s}\cos(ys)}{s + \sqrt{s^2 + \omega^2\mu_0\epsilon_0(1 - \mu_r\epsilon_r) + j\omega\mu_e\sigma_e}} \tag{11}$$

$$M + jN = 2\int_0^\infty F_2(s)\,ds \tag{12}$$

$$F_2(s) = \frac{e^{-(h_i+h_j)s}\cos(ys)}{\sqrt{s^2 + \omega^2\mu_0\epsilon_0(1 - \mu_r\epsilon_r) + j\omega\mu_e\sigma_e} + \frac{(\sigma_e + j\omega\epsilon_e)s}{j\omega\epsilon_0}} \tag{13}$$

Where, μ_0 is the vacuum permeability, μ_r is the relative permeability, ϵ_0 is the vacuum permittivity, ϵ_r is the relative permittivity.

The earth return admittance can be calculated as $Y_{eij} = j\omega[C], [C] = [P_{ij}]^{-1}$. Note for the case of only a single conductor where $i = j$, $y =$ radius of conductor.

One difficulty associated with the overhead line equations is the evaluation of the infinite integral. For any numerical integration, two important parameters need to be determined are point of truncation and discretization step.

Often the improper selection of truncation point will lead to improper discretization step, and thus erroneous earth-return parameters are computed.

Usually, the selection of truncation point can be roughly estimated by considering the exponential decay term in (11) and (13). A logical choice is to select the truncation point as $\frac{-\ln 0.001}{h_i+h_j}$, which corresponds to a 99.9% magnitude decay of initial value (i.e. $f(s = 0)$). Next, for a given discretization interval, the discretization step can be determined. A simplistic approach is to use a recursive method such that discretization step is decremented during each iteration until a convergent solution is obtained. However, this method can be numerically expensive. The number of samples required for the earth-return parameters to converge tend to be unreasonably high for cases with extreme frequency and long line separation.

An example is shown by calculating the earth return parameters of overhead-line arrangement shown in Fig. 1. The accurate earth return values are computed for the input parameters tabulated below.

TABLE I. TRANSMISSION LINE DATA

h_1, h_2	$[5,10,20,50,100,150,200]\,meters$
y	$[10^{-3}, 10^{-2}, 10^{-1}, 1, 5, 10, 100, 1000\,]\,meters$
μ_r	$[1, 5, 10]$
ϵ_r	$[1, 5, 10]$
σ_e	$[10^{-1}, 2\times10^{-2}, 10^{-3}, 5\times10^{-4}, 3.3\times10^{-4}, 2.5\times10^{-4}]\,\frac{1}{\Omega m}$
f	$\begin{bmatrix}10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1} \\ 10^{0}, 10^{1}, 10^{2}, 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}, 10^{8}\end{bmatrix} Hz$

The earth return values for 63,504 input parameter combinations using table I are calculated. A percentage error analysis of calculated earth return parameter values vs converged earth return parameter values are tabulated in Table II. Error is defined as when any of the calculated values (i.e. R,L,G,C) showed more than 10% difference in magnitude comparing to the converged values. As a summary, a recursive solver as ones shown below with a logical initial guess for truncation will still require many iterations prior to obtaining an acceptable solution for majority of the cases in Table I.

TABLE II. CONVERGENCE VS DISCRETIZATION

Number of samples	5×10^1	5×10^2	5×10^3	5×10^4
Percentage Error	78.97%	73.55%	60.07%	33.875%

In order to efficiently discretize and perform integration for the extended TL equations, a systematic procedure is developed analytically and described in the section III.

III. Analytical break down of Extended TL equation

A. Analysis of error

The extended TL equations can be interpreted as two exponential decaying sinusoidal (i.e. one for impedance, one for admittance). This is further separated into a message term and three envelope terms as shown below:

$$\mathrm{f_{msg}}(s) = \cos(ys) \tag{14}$$

$$\mathrm{f_{ev1}}(s) = \mathrm{e}^{-(h_i+h_j)s} \tag{15}$$

$$\mathrm{f^z_{ev2}}(s) = \frac{1}{s + \sqrt{s^2 + \omega^2\mu_0\epsilon_0(1-\mu_r\epsilon_r) + \mathrm{j}\omega\mu_e\sigma_e}} \tag{16}$$

$$\mathrm{f^y_{ev2}}(s) = \frac{1}{\dfrac{(\sigma_e + \mathrm{j}\omega\epsilon_e)s}{\mathrm{j}\omega\epsilon_0} + \sqrt{s^2 + \omega^2\mu_0\epsilon_0(1-\mu_r\epsilon_r) + \mathrm{j}\omega\mu_e\sigma_e}} \tag{17}$$

Where, f^z_{ev2} *and* f^y_{ev2} represent the complex square root envelope term in (11), and (13) respectively. An important property of the envelopes is $\lim_{s\to\infty} f_{ev1}(s) = 0$, and $\lim_{s\to\infty} f_{ev2}(s) = 0$. Hence, with proper discretization, the convergence of the integration is guaranteed.

As shown in the previous section, the increase in discretization frequency leads to reduction of calculation error for the resulting earth-return values. This is due to the exponential decay nature of the line equations themselves. The error in the integration can be explained using the following example. The real and imaginary parts of (11), and (13) using line parameters of $h_1, h_2 = 100, y = 100, \mu_r = 1, \epsilon_r = 10, \sigma_e = 2.5 \times 10^{-4}, f = 0.001Hz$ are shown in Fig. 2.

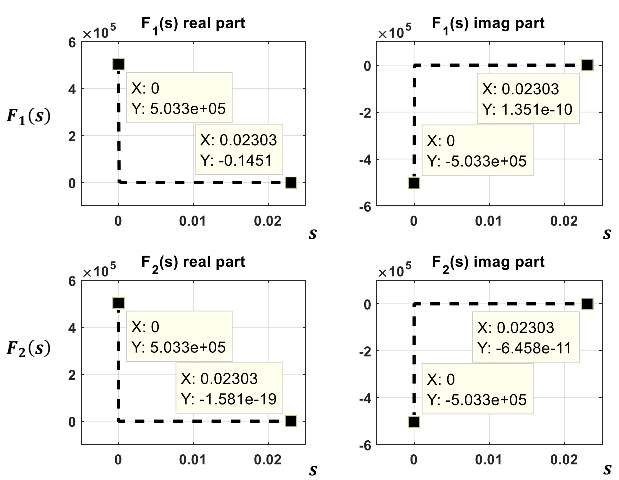

Figure 2. Real and Imaginary part of earth-return integrand

It can be observed that the majority of the integrand information is retained in the first 1% of the discretization (x axis). This is due to the complex square root envelope of (16), and (17) having higher a decay effect than the exponential decay term of (15). Therefore, grossly erroneous result will be obtained when discretization is solely based on the exponential decay term.

To further clarify the problem (16), (17) is extracted from Fig. 2 and plotted in Fig. 3. It can be observed in Fig. 3 that the real and imaginary component of (16), and (17) have widely different behaviors (i.e. maxima, magnitude, magnitude polarity change). Hence, a truncation point should not be generalized for (7), and (8). From this example, it can be deduced that the improper integration is due to:

1) Selection of discretization parameter without considering both real and imaginary parts.

2) Neglecting the effects of the square root envelope in (15), and (16). (i.e. magnitude, polarity change).

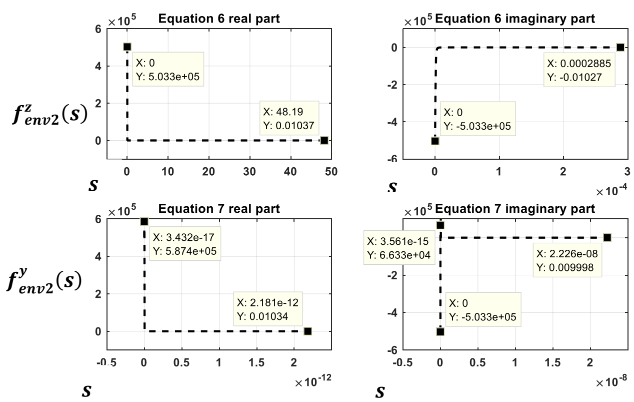

Figure 3. Real and imaginary part of the Eq. (6), and (7)

B. The solution procedure

To evaluate the extended TL earth return equations effectively, the point of truncation and discretization step for integration needs to be calculated analytically and assessed logically. As concluded in the previous section, the integration variables need to be determined for both real and imaginary part of (6), and (7). In order to do this one needs to:

1) Isolate the real/imaginary component from (16), (17).
2) Determine the equation maxima and its overall effect.
3) Determine the equation frequency and its overall effect.

1) Isolate the real/imaginary component

Using mathematical manipulation (16), and (17) are split into their corresponding real and imaginary parts as shown below.

$$\mathrm{real}(\mathrm{f^z_{ev2}}(s)) = \frac{s + \alpha}{s^2 + \alpha^2 + 2\alpha s + \beta^2} \tag{18}$$

$$\mathrm{imag}\left(f_{ev2}^{z}(s)\right) = \frac{-\beta}{s^2 + \alpha^2 + 2\alpha s + \beta^2} \quad (19)$$

$$\mathrm{real}\left(f_{ev2}^{y}(s)\right) = \frac{as + \alpha}{(a^2 + b^2)s^2 + 2(a\alpha + b\beta)s + \alpha^2 + \beta^2} \quad (20)$$

$$\mathrm{imag}\left(f_{ev2}^{y}(s)\right) = \frac{-bs - \beta}{(a^2 + b^2)s^2 + 2(a\alpha + b\beta)s + \alpha^2 + \beta^2} \quad (21)$$

Where:

$$\alpha = \sqrt{0.5}\sqrt{\sqrt{(s^2 + k_1)^2 + k_2^2} + (s^2 + k_1)}, a = \mu_r\epsilon_r \quad (22)$$

$$\beta = \sqrt{0.5}\sqrt{\sqrt{(s^2 + k_1)^2 + k_2^2} - (s^2 + k_1)}, b = -\frac{\mu_r\sigma_e}{\omega\epsilon_0} \quad (23)$$

$$k_1 = \omega^2\mu_0\epsilon_0(1 - \mu_r\epsilon_r), k_2 = \omega\mu_e\sigma_e \quad (24)$$

Now that integral is divided into real and imaginary components, each individual envelope's impact on integral decay can be assessed. This information is then used to determine the point of truncation. This can be described using the logic shown in (25).

$$P_{truncation} = \min(pt1, pt2) \quad (25)$$

Where, pt1, and pt2 each represents the 99.9% magnitude decay of the exponential decay and complex square root decay respectively. The analytical solution of pt1 is shown below,

$$pt1 = \frac{-\ln 0.001}{h_i + h_j} \quad (26)$$

2) Determine equation maxima and its overall effect

The evaluation of pt2 is not as straightforward since the effect of maxima also needs to be determined. This can be demonstrated using the imaginary part of (17) as shown in Fig. 3. If truncation was determined directly by solving position s such that $f_{env2}^{y}(s) = \frac{\mathrm{abs}(f(s=0))}{f(s=0)}0.001$, then the resulting truncation point will occur prior to the envelope maxima. This will produce a large truncation error due to early truncation. Therefore, the maxima need to first be identified and then used to determine the truncation point (i.e. solve for $f_{env2}^{y} = \frac{\mathrm{abs}(f_{maxima}(s))}{f_{maxima}(s)}0.001$).

In this paper, the maximas are found numerically using the golden section search [24]. Other numerical methods can also be adopted.

Golden section search requires an initial lower and upper bound. While the initial lower bound can be directly taken as the initial point of corresponding integral (i.e. $f(s = 0)$), the initial upper bound needs to be solved. Using (20) as an example, the upper bound can be solved using (27) by equating k_3 to the initial point of the integral. This is equivalent of finding when the initial point will reoccur down the integrand path. Note this needs to be performed for (18),(19),(21) as well in similar manner.

$$\frac{as + \alpha}{(a^2 + b^2)s^2 + 2(a\alpha + b\beta)s + \alpha^2 + \beta^2} - k3 = 0 \quad (27)$$

The analytical solution of (27) is obtained as (28) through mathematical manipulation. This also needs to be solved for (18), (19), (21), the solution is attached in appendix.

The root of the (28) can be found as the eigenvalue of its companion matrix [25]. It should be noted that (28) has 8 sets of eigenvalues. Therefore, one will need to take all the positive real ones and substitute back into (20) to determine if it is the appropriate one.

$$\begin{aligned} 0 = (L_1^2)s^8 + (2L_1L_2)s^7 + (2L_1L_3 + L_2L_2)s^6 \\ + (2L_1L_4 + 2L_2L_3)s^5 \\ + (2L_1k_4 + 2L_2L_4 + L_3L_3 \\ - 0.25)s^4 \\ + (L_2k_4 + 2L_3L_4 + k_4L_2)s^3 \\ + (2L_3k_4 + L_4L_4 - 0.5k_1)s^2 \\ + (2L_4k_4)s + k_4k_4 \\ - 0.25(k_1k_1 + k_2k_2) \end{aligned} \quad (28)$$

where,

$$\begin{aligned} L_1 &= 2a^2k_3^2 - 2b^2k_3^2 - k_3^2a^4 - k_3^2b^4 - 2k_3^2a^2b^2 - k_3^2 \\ L_2 &= -2ak_3 + 2k_3a^3 + 2k_3ab^2 \\ L_3 &= 0.5 + 2a^2k_3^2k_1 - 2b^2k_3^2k_1 + 4abk_3^2k_2 - 2k_3^2k_1 - a^2 \\ L_4 &= -2ak_3k_1 - 2bk_3k_2 \\ k_4 &= 0.5k_1 - k_1^2k_3^2 - k_2^2k_3^2 \end{aligned}$$

After obtaining the maxima, the magnitude at truncation point is determined as $k_3 = \frac{\mathrm{abs}(f(s=\max))}{f(s=\max)}0.001$. pt2 can then be solved by applying the new k_3 to (26) . This will yield the pt2 for (20). The pt2 will need to be calculated for (18), (19), (21) in a similar manner.

3) Determine the equation frequency and its overall effect.

With the truncation point determined, one remaining variable that needs to be solved is the discretization step. The integrand of the earth return equations has either an exponential decay characteristic, or an even faster decaying characteristic with an initial overshoot. Therefore, the use of logarithmic discretization is much appropriate in this scenario. Logarithmic discretization is applied by splitting the discretization axis into serval decades and then apply equal distance discretization to each decade separately. However, a special case arises where the message term of (14) has a much higher oscillation period then the calculated logarithmic discretization step. This can be demonstrated in Fig. 4.

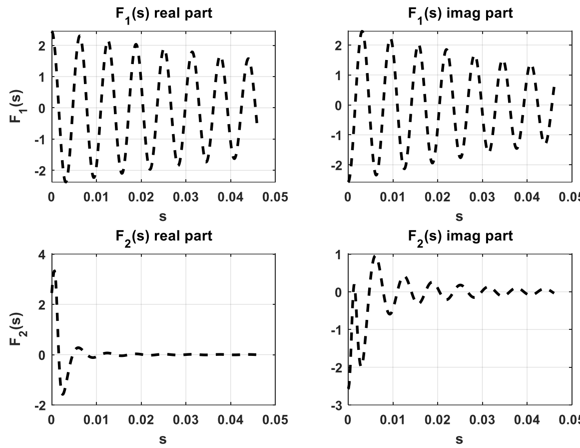

Figure 4. Real and imaginary part of the earth-return integrand

Fig. 4 is constructed using transmission line parameters of: $h_1, h_2 = 5, y = 1000, \mu_r = 1, \epsilon_r = 10, \sigma_e = 0.01, f = 1MHz$. It shows the complete integrand of (7),(8) described by (11), (13). It is observed that the oscillatory component has a dominant effect over the entire discretization interval. Hence, if the discretization step was calculated to be smaller than the oscillation period of (14) then Shannon's sampling theorem is violated and integral will yield erroneous results. Therefore, an upper bound needs to be enforced for logarithmic discretization step calculation. The following logic is used in this paper to enforce the upper bound.

$$\mathrm{ds} = \min\left(\mathrm{P_{decade}}, \frac{\pi}{5\mathrm{y}}\right) \tag{29}$$

Where ds is the discretization step and $\mathrm{P_{decade}}$ is the calculated discretization step. (29) enforces discretization step such that it cannot be larger than 10% of the message term's oscillatory period.

C. Summary of the solution

This concludes the calculation of all discretization parameters needed. A chronological procedure to discretize the extended TL formula is shown below.

1) Determine 4 truncation point pt1 due to exponential term using (26).
2) Split (16), (17) into real and imaginary parts respectively using (18), (19), (20), and (21).
3) Determine the maxima and its effect for (18), (20), (21) using the analytical solution (28), and respective ones from appendix. You do not need to do this for (19) as it does not have a maxima.
4) Determine the 4 $\mathrm{k_3}$ using the found maxima. If maxima does not exist use $k_3 = f(s = 0)$
5) Solve for the 4 pt2s by applying the 4 $\mathrm{k_3}$ obtained in step 4. To the analytical solutions again in step 3.
6) Determine the 4 truncation points based on the 4 pt1 from step 1 and 4 pt2 from step 5.
7) Determine the logarithmic discretization axis and enforce the upper bound of (29). This paper adopted $ceil\left(\frac{maxima}{0.01}\right)$ decades for each integral.
8) Solve the infinite integral using any numerical method of choice.

The numerical integration of the extended TL equation is straightforward now that both discretization interval and discretization axis confirmed. Also, since both parameters are analytical/numerically derived there will be no need for iterative step to ensure correct convergence.

IV. Validation

For validation, Fig. 2 is numerically integrated using equal distance discretization and the proposed integration technique. The results are plotted in Fig. 5.

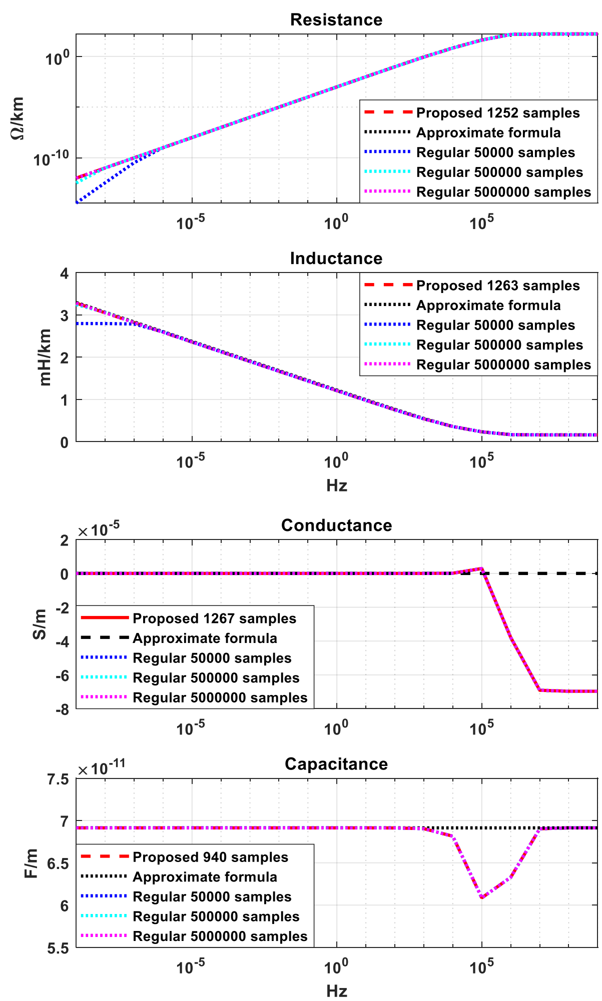

Figure 5. Earth return parameters vs frequency

The proposed method used 9~11 decade for logarithmic integration for a total of maximum 1267 discretization points. While on the other hand, equal distance discretization used 50,000 to 5,000,000 discretization point per integral. integration results suggest that the proposed method will always yield a more accurate solution using fewer amount of discretization points. It should be noted even if logarithmic sampling is not used, adopting the calculated truncation point will significantly improve the accuracy of the equal distance discretization method.

A benefit of the extended TL earth return formula vs approximate TL earth return formula can also be observed. The approximate formula used is based on modified Carson's formula to take in the consideration of displacement current [17]. As shown in the frequency response, the high frequency mode transition phenomena past 100kHz can be characterized by the extended formula [17].

V. Conclusions

In this paper, a numerically efficient integration technique to evaluate the extended TL earth return formula is developed. Its analytical formulations and chronological procedures are described. Discretization parameters are computed based on the impact of the real/imaginary part of the earth-return equations. As a result, the proposed technique was able to compute the earth-return equation using much fewer discretization points while yielding highly accurate results.

VI. Appendix

(9), (10), (11), (12) are expanded from (7), and (8) using the following relation. Given any complex square root with a real part of x_1, and imaginary part of x_2, its solution can be computed as $x_3 + jx_4$.

$$\sqrt{x_1 + jx_2} = x_3 + jx_4$$

$$x_3 = 0.707\sqrt{x_1 + \sqrt{x_1^2 + x_2^2}}$$

$$x_4 = 0.707\sqrt{\sqrt{x_1^2 + x_2^2} - x_1}$$

Analytical solution for (18).

$$0 = 2\mathrm{L}_3\mathrm{L}_4\mathrm{s}^3 + (2\mathrm{L}_3\mathrm{k}_4 + \mathrm{L}_4^2 - 0.5\mathrm{k}_1)\mathrm{s}^2 + 2\mathrm{L}_4\mathrm{k}_4\mathrm{s} + \mathrm{k}_4^2 - 0.25\mathrm{k}_1^2 - 0.25\mathrm{k}_2^2$$

Where:

$\mathrm{L}_3 = -0.5$

$\mathrm{L}_4 = 2\mathrm{k}_2\mathrm{k}_3$

$\mathrm{k}_4 = -0.5\mathrm{k}_1 - \mathrm{k}_3^2\mathrm{k}_1^2 - \mathrm{k}_3^2\mathrm{k}_2^2$

Analytical solution for (19).

$$0 = 2\mathrm{L}_3\mathrm{L}_4\mathrm{s}^3 + (2\mathrm{L}_3\mathrm{k}_4 + \mathrm{L}_4^2 - 0.5\mathrm{k}_1)\mathrm{s}^2 + 2\mathrm{L}_4\mathrm{k}_4\mathrm{s} + \mathrm{k}_4^2 - 0.25\mathrm{k}_1^2 - 0.25\mathrm{k}_2^2$$

Where:

$\mathrm{L}_3 = -0.5$

$\mathrm{L}_4 = -2\mathrm{k}_1\mathrm{k}_3$

$\mathrm{k}_4 = -0.5\mathrm{k}_1 - \mathrm{k}_3^2\mathrm{k}_1^2 - \mathrm{k}_3^2\mathrm{k}_2^2$

Analytical solution for (21).

$$0 = (\mathrm{L}_1^2)\mathrm{s}^8 + (2\mathrm{L}_1\mathrm{L}_2)\mathrm{s}^7 + (2\mathrm{L}_1\mathrm{L}_3 + \mathrm{L}_2\mathrm{L}_2)\mathrm{s}^6 + (2\mathrm{L}_1\mathrm{L}_4 + 2\mathrm{L}_2\mathrm{L}_3)\mathrm{s}^5 + (2\mathrm{L}_1\mathrm{k}_4 + 2\mathrm{L}_2\mathrm{L}_4 + \mathrm{L}_3\mathrm{L}_3 - 0.25)\mathrm{s}^4 + (\mathrm{L}_2\mathrm{k}_4 + 2\mathrm{L}_3\mathrm{L}_4 + \mathrm{k}_4\mathrm{L}_2)\mathrm{s}^3 + (2\mathrm{L}_3\mathrm{k}_4 + \mathrm{L}_4\mathrm{L}_4 - 0.5\mathrm{k}_1)\mathrm{s}^2 + (2\mathrm{L}_4\mathrm{k}_4)\mathrm{s} + \mathrm{k}_4\mathrm{k}_4 - 0.25(\mathrm{k}_1\mathrm{k}_1 + \mathrm{k}_2\mathrm{k}_2)$$

Where:

$\mathrm{L}_1 = 2\mathrm{a}^2\mathrm{k}_3^2 - 2\mathrm{b}^2\mathrm{k}_3^2 - \mathrm{k}_3^2\mathrm{a}^4 - \mathrm{k}_3^2\mathrm{b}^4 - 2\mathrm{k}_3^2\mathrm{a}^2\mathrm{b}^2 - \mathrm{k}_3^2$

$\mathrm{L}_2 = -2\mathrm{b}\mathrm{k}_3 - 2\mathrm{k}_3\mathrm{a}^2\mathrm{b} - 2\mathrm{k}_3\mathrm{b}^3$

$\mathrm{L}_3 = 2\mathrm{a}^2\mathrm{k}_1\mathrm{k}_3^2 - 0.5 - 2\mathrm{b}^2\mathrm{k}_1\mathrm{k}_3^2 + 4\mathrm{a}\mathrm{b}\mathrm{k}_2\mathrm{k}_3^2 - 2\mathrm{k}_3^2\mathrm{k}_1 - \mathrm{b}^2$

$\mathrm{L}_4 = 2\mathrm{a}\mathrm{k}_2\mathrm{k}_3 - 2\mathrm{b}\mathrm{k}_1\mathrm{k}_3$

$\mathrm{k}_4 = -0.5\mathrm{k}_1 - \mathrm{k}_1^2\mathrm{k}_3^2 + \mathrm{k}_2^2\mathrm{k}_3^2$

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