Higher-Order Newton-Cotes and Gauss-Quadrature Integration Rules to Solve Carson and Pollaczek Integrals

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***Abstract*-- The problem of electromagnetic wave propagation, despite its importance, does not have a simple analytical solution. Therefore, approximations—such as assuming a semi-infinite homogeneous ground plane—are of practical interest. Using quasi-static approximations, Carson and Pollaczek derived integral equations to calculate the electromagnetic field produced by a horizontal current over a lossy ground plane. Carson proposed the first solution to these expressions using power series expansions, but this approach lacks uniform convergence. Since then, various efforts have been made to obtain more accurate solutions. In this sense, main approaches have been developed to solve these integral equations. The first involves modifying the integrand to obtain an analytic solution, while the second relies on using numerical integration techniques, which are essential for solving integrals without closed-form solutions. The accuracy of numerical methods is influenced by the choice of integration scheme, order, and number of samples. However, theoretical expectations of improved accuracy with higher-order methods and increased sample points are often limited by numerical representation constraints. One of this constrain is the finite bit representation in binary calculations. Despite these limitations, numerical methods remain the only viable approach for certain integrals. This work presents the implementation of Newton and Gauss integration methods, analyzing their performance concerning method type, order, and sample count. Since Carson and Pollaczek's equations include a decreasing exponential term, the infinite upper limit is replaced by a finite bound without exceeding a predefined error threshold. By applying this limit substitution and numerical techniques, we obtain a new solution that ensures uniform convergence across all cases.**

***Keywords*: Carson Integrals, Pollaczek Integrals, Numerical Integration Methods, Newton-Cotes Methods, Gauss-Quadrature Methods, Higher-Order Integration Methods.**

I. Introduction

In the area of electrical power systems there are two transient phenomena that involve integrals that lack analytical solutions. The first of these is known as the Carson integral [1], which models the behavior of the electromagnetic field on the ground due to the flow of current in an overhead line. The second is the Pollaczek integral [2], which models the distribution of the electromagnetic field in the underground due to the flow of current in an underground cable. The first approximation to the Carson integral was made by Carson using power series [1]. Since then, several solutions for this integral have been proposed: a) by replacing the integrand with functions that behave in a similar way, but that exhibit similar behavior but have known analytical solutions [3-15], b) using different types of series that provide solutions to the integral [1, 16, 17], and c) applying numerical methods [18-21]. Similarly, for the Pollaczek integral, approximations have been made by changing functions to more manageable forms for analysis [21-26], and numerical methods have also been applied [28-35]. These integrals present two main challenges: a) an infinite upper limit, and b) variable parameters that depend on factors such as geometry, frequency, medium composition, and conductor height. This results in virtually infinite possible configurations, meaning that each specific case presents a unique integral to solve. Additionally, when the problem depends on frequency, the number of integrals to solve increases proportionally to the number of frequencies involved in the study.

It is important to address that the use of numerical methods has not been fully adapted or studied for solving these integrals due to the challenges in calculating the coefficients of high-order numerical integration formulas [36]. The difficulty in calculating the coefficients for both Gaussian and Newtonian formulations has led to the predominance of low-order formulas until recent years. However, with the advancement of computing technology, higher-order formulas can now be explored. Despite this capability, a thorough investigation has yet to be conducted to determine whether increasing the number of data points and the order of the formulas improves the results or causes degradation. This is the primary objective of the present work.

Therefore, this article presents a study of high-order Newton formulas (both Open and Closed Newton Formulas) and Gauss-Legendre Quadrature. For Newton's Open formulas, an order of 33 was achieved, as higher orders led to computational errors due to limitations in 64-bit computing. The same issue was encountered with Newton's Closed formulas at order 33. For Gauss-Legendre Quadrature, coefficients can be calculated up to very high orders (e.g., 200 or more). However, beyond a certain point, the solutions began to degrade instead of improving, so only the first 33 orders were implemented in this study.

II. Numerical Methods

Integration methods can be evaluated based on several algebraic characteristics that indicate their proper development: 1) the coefficients should exhibit symmetry from the ends toward the center, 2) the coefficients should be rational numbers, and 3) the sum of coefficients is an integer equal to

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the number of intervals used in their derivation.

In the case of Gauss-Quadrature, the sum of coefficients is always 2 because the method is derived from integrating over the interval from -1 to +1, which has a length of 2. However, as high-order methods are developed, there is a risk of violating these algebraic principles, which can lead to inaccuracies in the numerical solutions.

In this section we outline the algebraic procedure for generating all the aforementioned methods. As an example, the 14th-order method derived from this procedure is presented to demonstrate how these methods exhibit the previously described algebraic properties. Furthermore, a function with behavior similar to Carson's and Pollaczek's integrals is numerically integrated using all the generated methods. A numerical study is performed on this integral to verify whether increasing the number of samples improves accuracy and whether higher-order methods achieve greater precision.

A. Closed Newton-Cotes formulas

Assuming we aim to approximate the definite integral of a function in the interval $[a,b]$, one method to calculate this approximation consists of using the Newton interpolating polynomial with forward differences. For this type of formula, we have that $h=(b-a)/n$ and $x_k=a+kh,$ with $k=0,\ldots,n$. By notation, for closed formulas it is considered that $a=x_0$ and $b=x_n$. For Newton's forward differences interpolating polynomial with equally spaced data points between the know points $(x_0,f(x_0)),\ldots,(x_n,f(x_n))$, the formula is:

$$f(x)\approx\sum_{k=0}^{n}\Delta^k f_0\binom{s}{k}, \tag{1}$$

with $x=x_0+sh$ and $\binom{s}{k}=\frac{s(s-1)\ldots(s-k+1)}{k!}$. By integrating both sides of (1) from $a=x_0$ to $b=x_n$, and considering the change of variable used, we obtain:

$$\int_{x_0}^{x_n}f(x)dx\approx\int_0^n h\sum_{k=0}^{n}\Delta^k f_0\binom{s}{k}ds \tag{2}$$

Defining $b_{nk}=\int_0^n\binom{s}{k}ds$ and substituting into (2), we obtain the formulation of the closed Newton-Cotes as:

$$\int_{x_0}^{x_n}f(x)dx\approx h\sum_{k=0}^{n}\Delta^k f_0 b_{nk} \tag{3}$$

The development of high-order integration formulas is a repetitive process that encounters a numerical constraint due to the division by the term *k*!, where *k* increases up to *n*, the order of the formulation. As the order increases, computational limitations become evident, reaching a point where exact calculations of the coefficients involved in the formula are no longer feasible, and accuracy of the resulting integral is compromised. The resulting formula derived from this process when applied to the closed Newton formula of order 14 is presented in (4).

$$\begin{aligned}I_{14}^{closed}=h\Big(&\tfrac{1017}{4028}f_0+\tfrac{1108}{557}f_1-\tfrac{2579}{1196}f_2+\tfrac{3899}{398}f_3-\tfrac{5153}{278}f_4\\&+\tfrac{3145}{89}f_5-\tfrac{4654}{99}f_6+\tfrac{4427}{81}f_7-\tfrac{4654}{99}f_8+\tfrac{3145}{89}f_9\\&-\tfrac{5153}{278}f_{10}+\tfrac{3899}{398}f_{11}-\tfrac{2579}{1196}f_{12}+\tfrac{1108}{557}f_{13}+\tfrac{1017}{4028}f_{14}\Big)\end{aligned} \tag{4}$$

B. Open Newton-Cotes formulas

To approximate a definite integral of a function in the interval $[a,b]$. For this type of formulas, $h=(b-a)/(n+2)$ and $x_k=a+kh,$ with $k=0,\ldots,n+1$ are taken. For open formulas, by notation, $x_{-1}=a$ and $x_{n+1}=b$ are considered. Considering the Newton interpolating polynomial with advanced differences for equally spaced points that passes through the points $(x_{-1},f(x_{-1})),\ldots,(x_{n+1},f(x_{n+1}))$ we have $f(x)\approx\sum_{k=0}^{n}\Delta^k f_0\binom{s}{k}$ with $x=x_0+sh$. Integrating both sides from $a=x_{-1}$ to $b=x_{n+1}$ and taking into account the change of variable used, we obtain

$$\int_{x_{-1}}^{x_{n+1}}f(x)dx\approx\int_{-1}^{n+1}h\sum_{k=0}^{n}\Delta^k f_0\binom{s}{k}ds \tag{5}$$

Defining $c_{nk}=\int_{-1}^{n+1}\binom{s}{k}ds$ and substituting in (5), we obtain the formulation of the open Newton-Cotes formulas as:

$$\int_{x_{-1}}^{x_{n+1}}f(x)dx\approx h\sum_{k=0}^{n}\Delta^k f_0 c_{nk} \tag{6}$$

The process is similar to that of closed formulas, the one of order 14 for example is:

$$\begin{aligned}I_{14}^{open}=h\Big(&\tfrac{1490}{247}f_0-\tfrac{3381}{104}f_1+\tfrac{20315}{134}f_2-\tfrac{25921}{54}f_3+\tfrac{42351}{37}f_4\\&-\tfrac{77127}{37}f_5+\tfrac{44581}{15}f_6-\tfrac{100157}{30}f_7+\tfrac{44581}{15}f_8-\tfrac{77127}{37}f_9\\&+\tfrac{42351}{37}f_{10}-\tfrac{25921}{54}f_{11}+\tfrac{20315}{134}f_{12}-\tfrac{3381}{104}f_{13}+\tfrac{1490}{247}f_{14}\Big)\end{aligned} \tag{7}$$

C. Gauss-Legendre Quadrature

The formula for the approximation of an integral is defined:

$$\int_{-1}^{+1}f(x)dx=\sum_{i=0}^{n}a_i\cdot f(x_i)+E, \tag{8}$$

where x_i are the roots of the Legendre polynomial used to approximate the integral. To approximate the integral by a Gaussian-Legendre Quadrature method, the orthogonal basis $f(x)=1,\ x,\ x^2,\ x^3,\ldots,x^n$ and the region of orthogonally of the Legendre polynomials is used.

By analyzing the process of obtaining the Gauss-Legendre Quadrature formulations, a general procedure can be obtained for any order. For an approximation of α terms, we start from the roots of the Legendre polynomial $P^{\alpha}(x)$ and a system of equations of $\alpha\times\alpha$. For example, if we have that $\alpha=2$, we arrive at the following system of equations:

$$2=a_0\cdot x_0^0+a_1\cdot x_1^0+a_2\cdot x_2^0 \tag{9a}$$

$$0=a_0\cdot x_0^1+a_1\cdot x_1^1+a_2\cdot x_2^1 \tag{9b}$$

$$\frac{2}{3} = a_0 \cdot x_0^2 + a_1 \cdot x_1^2 + a_2 \cdot x_2^2 \tag{9c}$$

In matrix form we have:

$$\begin{bmatrix} x_0^0 & x_1^0 & x_2^0 \\ x_0^1 & x_1^1 & x_2^1 \\ x_0^2 & x_1^2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix} \tag{10}$$

The general form of this system of equations is with $\alpha = n$, from which we obtain:

$$\begin{bmatrix} x_0^0 & x_1^0 & x_2^0 & \cdots & x_n^0 \\ x_0^1 & x_1^1 & x_2^1 & \cdots & x_n^1 \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \\ \vdots \\ 0 \text{ ó } \frac{2}{n+1} \end{bmatrix}, \tag{11}$$

where x_n are the roots of the nth Legendre polynomial.

In this way it can be easily noted that the only restriction on the order of the approximation is the calculation of the inverse of the matrix. To make the change at a general interval, we must have,

$$\int_a^b f(z)\,dz = \int_{-1}^{+1} f(z)\,(dz/dx)dx = \frac{b-a}{2}\sum_{k=0}^{N} w_k f(z_k), \tag{12}$$

where $\frac{dz}{dx} = \frac{b-a}{2}$, $z_k = \frac{(b-a)x_k + b + a}{2}$ and $w_k = a_k$.

The system of equations is formed to calculate the coefficients of the formulation, thus obtaining the formula for the Gauss-Legendre Quadrature of order 14,

$$\begin{aligned} I_{14}^{GLQ} = \frac{h}{2}\Big(& \tfrac{296}{9625} f_0 + \tfrac{421}{5983} f_1 + \tfrac{235}{2193} f_2 + \tfrac{91}{652} f_3 + \tfrac{279}{1678} f_4 \\ & + \tfrac{643}{3454} f_5 + \tfrac{253}{1275} f_6 + \tfrac{110}{543} f_7 + \tfrac{253}{1275} f_8 + \tfrac{643}{3454} f_9 \\ & + \tfrac{279}{1678} f_{10} + \tfrac{91}{652} f_{11} + \tfrac{235}{2193} f_{12} + \tfrac{421}{5983} f_{13} + \tfrac{296}{9625} f_{14} \Big) \end{aligned} \tag{13}$$

D. Numerical Example

This section shows the results of all the formulas for the following function:

$$f(t) = \int_0^{2\pi} t^2 \cos(t) e^{-0.01t}\,dt.$$

In green, the five formulas are selected (the one with the highest order is also selected to analyze its behavior) for the presented cases; the selection is made based on the maximum error knowing that it starts with 800 samples in steps of 100 up to 10000 samples. It is selected with the maximum error because an adequate number of samples cannot be selected a priori, so regardless of the number of samples (greater than 800 samples), there will be an error that is between the maximum and the minimum. The results from 4 examples, which are in the public repository with the link: https://docs.google.com/document/d/1iprolV2vdReGK7lYvFKIOQo2OwDzQoHw/edit.

TABLE I
MAXIMUM PERCENT DIFFERENCE OF NEWTON CLOSE FORMULAS, NEWTON OPEN FORMULAS AND GAUSS QUADRATURE FORMULAS

CLOSE NEWTON FORMULA				
Orden	*Error min*	*Samples*	*Error max*	*Samples*
1	3.7604e-07	10000	5.8757e-05	800
2	4.6185e-14	9600	7.2476e-10	800
3	8.7041e-14	9800	1.6226e-09	800
4	7.1054e-15	800	2.6645e-14	6600
5	3.5527e-15	1000	4.0856e-14	800
6	0	5200	1.7764e-14	9900
7	1.7764e-15	3200	1.4211e-14	2300
8	0	1700	1.0658e-14	2900
9	2.3803e-13	800	2.558e-13	3600
10	1.2257e-13	2100	1.4566e-13	8200
11	2.7356e-13	5400	2.9488e-13	900
12	2.3981e-13	800	3.1442e-13	9600
13	1.1084e-11	800	1.1743e-11	9000
14	3.9801e-11	800	4.2233e-11	9400
15	2.1151e-11	9600	2.3142e-11	800
16	4.8342e-11	800	5.5151e-11	8200
17	3.2592e-10	800	3.3398e-10	9900
18	6.1739e-10	800	7.063e-10	9900
19	4.9429e-09	800	5.3217e-09	10000
20	2.5668e-09	800	3.7003e-09	10000
21	6.245e-08	800	6.9947e-08	10000
22	4.408e-08	800	4.619e-08	10000
23	1.5858e-07	800	1.8307e-07	10000
24	1.673e-07	800	1.8367e-07	10000
25	7.3644e-07	800	9.3529e-07	10000
26	1.0107e-05	800	1.156e-05	10000
27	4.2095e-06	800	6.0028e-06	10000
28	5.2403e-05	800	5.7101e-05	9900
29	1.632e-05	10000	2.5719e-05	800
30	0.00022325	800	0.00024463	10000
31	0.0005681	800	0.00058537	9700
32	0.00060684	800	0.0007625	9800
33	0.0046319	800	0.0054739	9800

OPEN NEWTON FORMULA				
Orden	*Error min*	*Samples*	*Error max*	*Samples*
0	7.5209e-07	10000	0.00011752	800
1	1.1284e-06	10000	0.00017584	800
2	4.0856e-13	10000	1.0147e-08	800
3	6.8923e-13	10000	1.7214e-08	800
4	1.7764e-15	5200	6.7502e-13	800
5	2.1316e-14	4600	1.2115e-12	800
6	0	6200	6.0396e-14	9600
7	2.8422e-14	7200	6.5725e-14	5200
8	0	1600	1.0125e-13	8500
9	2.7001e-13	9600	3.3573e-13	9900
10	4.885e-13	10000	1.3216e-12	9600
11	4.6505e-12	800	4.9027e-12	6300
12	1.5646e-11	7300	1.7502e-11	9300
13	1.1664e-11	800	1.3086e-11	7000
14	6.5317e-11	800	7.5561e-11	8200
15	1.0414e-10	800	1.1358e-10	6600
16	8.2971e-10	800	9.0085e-10	8600
17	2.1208e-10	800	2.4691e-10	8800
18	8.619e-10	8200	1.015e-09	800
19	5.9644e-09	10000	6.4218e-09	800
20	3.2217e-08	800	3.4039e-08	8400
21	2.5791e-08	800	3.4992e-08	9900
22	2.7376e-07	800	2.9664e-07	10000
23	1.0561e-06	800	1.2047e-06	10000
24	1.633e-07	800	2.359e-07	9800
25	3.3775e-06	9600	3.4093e-06	800
26	2.1861e-05	800	2.3536e-05	9900
27	4.623e-05	800	4.9048e-05	10000
28	3.3968e-05	800	3.6347e-05	9900
29	9.6978e-05	800	0.00010696	10000

30	0.00089168	800	0.00096582	10000
31	0.0050718	800	0.0056201	10000
32	0.0061996	800	0.007866	9900
33	0.027494	800	0.033993	9900

GAUSS LEGENDRE QUADRATURE				
Orden	*Error min*	*Samples*	*Error max*	*Samples*
1	0	5300	3.0237e-11	800
2	0	800	9.77e-14	9200
3	0	4000	8.5265e-14	8000
4	0	1600	7.4607e-14	9000
5	0	3200	6.7502e-14	4400
6	0	3800	5.6843e-14	3400
7	0	2800	5.6843e-14	9200
8	0	3800	6.0396e-14	9000
9	0	3600	6.2172e-14	8900
10	0	5700	7.1054e-14	8700
11	0	4400	5.862e-14	7100
12	0	4800	5.5067e-14	3600
13	0	2600	5.3291e-14	3900
14	0	4700	6.3949e-14	9700
15	0	1700	5.6843e-14	4500
16	0	1800	5.6843e-14	4800
17	0	6900	5.5067e-14	5100
18	0	7200	6.0396e-14	5400
19	0	3000	6.2172e-14	7300
20	0	1000	5.5067e-14	6000
21	1.7764e-15	2000	5.3291e-14	6800
22	0	6700	5.5067e-14	6600
23	0	6600	5.862e-14	8600
24	0	3900	6.0396e-14	7200
25	0	4100	6.3949e-14	9600
26	0	3800	5.6843e-14	7800
27	0	1700	5.5067e-14	8100
28	0	8000	5.5067e-14	8400
29	1.7764e-15	4200	5.862e-14	8700
30	0	5800	5.862e-14	9000
31	0	4500	5.5067e-14	9300
32	0	800	5.6843e-14	9600
33	0	5200	5.5067e-14	9900

III. Carson Integrals

Considering a uniform line (the line material and the surrounding dielectric are homogeneous) and neglecting the current displacement, the self and mutual earth impedance described the Carson integrals are [1]:

$$Z_{E,ii} = \frac{j\omega\mu}{2\pi}\int_0^{+\infty}\left(\sqrt{\alpha^2+j}-\alpha\right)e^{-2h'\alpha}d\alpha \tag{14a}$$

$$Z_{E,ik} = \frac{j\omega\mu}{2\pi}\int_0^{+\infty}\left(\sqrt{\alpha^2+j}-\alpha\right)e^{-(h_i'+h_k')\alpha}\cos(x'\alpha)\,d\alpha \tag{14b}$$

where μ is the air permeability; $h_i' = h_i\sqrt{\omega\mu\sigma}$ with h_i like the i-esime conductor height; $x' = x\sqrt{\omega\mu\sigma}$ with x like the distance between conductors; $\omega = 2\pi f$ with f like the frequency and σ is the earth conductivity. Defining,

$$J(p,q) = \int_0^{+\infty}\left(\sqrt{\alpha^2+j}-\alpha\right)e^{-p\alpha}\cos(q\alpha)\,d\alpha \tag{15}$$

for the self-impedance $p = 2h\phi$ and $q = 0$. For the mutual-impedance $p = h_i\phi + h_k\phi = (h_i + h_k)\phi$ and $q = x\phi$, being $\phi = \sqrt{\omega\mu\sigma}$. So, equation (14a) and equation (14b) could be,

$$Z_{E,ii} = \frac{j\omega\mu}{2\pi}J(2h\phi, 0) \tag{16a}$$

$$Z_{E,ik} = \frac{j\omega\mu}{2\pi}J\left((h_i + h_k)\phi, x\phi\right) \tag{16b}$$

A. Carson Series

Let it be $r = \sqrt{p^2+q^2}$ and $\theta = \tan^{-1}(q/p)$; Carson's integrals given by equation (15) could be separated in real and imaginary parts as [1], $J = P + jQ$,

$$P = \frac{1}{2}\left[\left(\ln\frac{2}{\gamma r}\right)s_2 + \frac{\pi}{4}(1-s_4) + \theta s_2' + \sigma_2\right] + \frac{1}{\sqrt{2}}(\sigma_3 - \sigma_1) \tag{17}$$

$$Q = \frac{1}{2}\left[\frac{1}{2} + \left(\ln\frac{2}{\gamma r}\right)(1-s_4) - \frac{\pi}{4}s_2 - \theta s_4' - \sigma_4\right] + \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3) \tag{18}$$

γ in equation (17) and equation (18), is the Euler's constant and the s_k, and σ_k are the Carson's series terms [1].

B. Approximate formulas

The process to obtain approximate formulas lies in the substitution of the term $\left(\sqrt{\alpha^2+j}-\alpha\right)$ inside the Carson's integral by a function with similar behavior with analytical solution. The proposed function by Gary is the following [4],

$$\left(\sqrt{\alpha^2+j}-\alpha\right) \approx \frac{j}{2\alpha}\left(1-e^{j2\alpha\sqrt{j}}\right) \tag{19}$$

By substituting the proposed function into equation (14a) and equation (14b), it is obtained

$$Z_{E,ii} = \frac{j\omega\mu}{2\pi}\int_0^{+\infty} j\left(1-e^{j2\alpha\sqrt{j}}\right)\frac{e^{-2h\alpha}}{2\alpha}d\alpha \tag{20a}$$

$$Z_{E,ik} = \frac{j\omega\mu}{2\pi}\int_0^{+\infty} j\left(1-e^{j2\alpha\sqrt{j}}\right)\frac{e^{-(h_i+h_k)\alpha}\cos(x\alpha)}{2\alpha}d\alpha \tag{20b}$$

These equations have a well-known analytical solution [4],

$$Z_{E,ii}^{Gary} = \frac{j\omega\mu}{2\pi}\ln\left(1+\frac{z}{h_i}\right) \tag{21a}$$

$$Z_{E,ik}^{Gary} = \frac{j\omega\mu}{2\pi}\left[\frac{1}{2}\ln\left(1+\frac{4zh_{ik}+4z^2}{D_{ik}^2}\right)\right] \tag{21b}$$

where $z = 1/(\phi\sqrt{j})$ and $D_{ik} = \sqrt{(h_i+h_k)^2+x^2}$.

So, the self and mutual impedance like a function of p and q are respectively,

$$Z_{E,ii}^{Gary} = \frac{j\omega\mu}{2\pi}\ln\left(1+\frac{2}{p\sqrt{j}}\right) \tag{22a}$$

$$Z_{E,ik}^{Gary} = \frac{j\omega\mu}{2\pi}\left\{\frac{1}{2}\ln\left[1+\left(\frac{4p}{\sqrt{j}}-j4\right)\Big/\left(p^2+q^2\right)\right]\right\} \tag{22b}$$

C. Qualitative Analysis

To realize the analysis of the Carson's integrals behavior, these are solved by the numerical integration methods. These solutions will be the point of reference, it is important to note

that the numerical implementation is a rigid solution in which one used millions of points if it is necessary to have the prescribed error, for this reason sometimes the process is slow but in every case one obtains a very trustful solution.

From the formulas, one could deduce that every single parameter makes that $\mathbf{Z}$ value change in different way, so to take into account the combined effect of these parameters, it is use the p and q variables of the Carson's integrals and all the combinations are referred to these variables. One specific combination of h_{ik}, x_{ik}, f and $\boldsymbol{\sigma}$ gives like a result some values of p and q; to take into account all possible variations inside the establish intervals according with equation (15), it is obtained the minimum and maximum values of p and q in the following way,

$$p_{\min} = h_{ik,\min}\sqrt{2\pi f_{\min}\mu\sigma_{\min}} \cong 1e^{-7} \tag{23a}$$

$$p_{\max} = h_{ik,\max}\sqrt{2\pi f_{\max}\mu\sigma_{\max}} \cong 1e^{4} \tag{23b}$$

The previous indicates that it could be possible to evaluate the effect of the variations of h_{ik}, x_{ik}, f and $\boldsymbol{\sigma}$ in all the establish interval by using the variation of p from their minimums to their maximums. To realize the analysis in base on p and q it could be notice that these values are connected each other because they share one term, which could be denote as,

$$g_O = \sqrt{2\pi f_O \mu\sigma_O} \tag{24}$$

Thus, if one has one p value, it could be calculated the limits as $p_x = h_{ik,\max}g_1$ and $p_x = h_{ik,\min}g_2$. So, one has $g_1 = p_x/h_{ik,\max}$ and $g_2 = p_x/h_{ik,\min}$. Because this value needs to be incorporated into q, their inferior and superior limits are,

$$q_{ini} = x_{ik,\min}g_1 = \left(x_{ik,\min}/h_{ik,\max}\right)p_x \tag{25}$$

$$q_{fin} = x_{ik,\max}g_2 = \left(x_{ik,\max}/h_{ik,\min}\right)p_x \tag{26}$$

in this way the interval of q variation for one specific p is

$$q = \left[x_{ik,\min}/h_{ik,\max} \quad x_{ik,\max}/h_{ik,\min}\right]p_x \tag{27}$$

Figure 1 shows the relationship between the parameter p and the parameter q; that is, they cannot vary independently since they depend directly on the combination of the four parameters h_{ik}, x_{ik}, f and $\boldsymbol{\sigma}$; in other words, all the variation of parameters is covered by changing p from a minimum value to a maximum value and for each case the minimum and maximum value of q is calculated with equation (27), in this way all possible scenarios are generated.

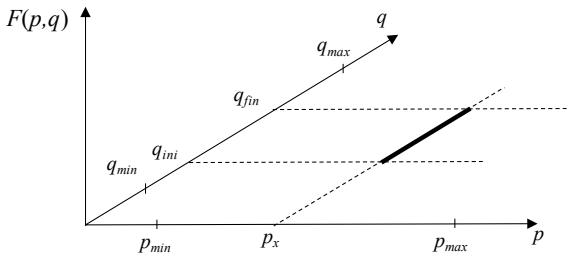

Fig. 1. Graph of where q begins and ends as a function of a specific p.

D. Superior limit for the Carson Integral

The Carson integral has zero as its lower limit, but infinite as its upper limit eq (15). To solve numerically, the upper limit is changed for each case making $p\alpha = 20$ ($e^{-p\alpha} \cong 0$), so the upper limit is $\alpha = 20/p$. If the minimum amplitude is 0.032224+j0.028634 units, then the maximum contribution occurs when p=5.0265, that is, α=3.9789. With this data as the lower limit and taking the upper limit as 20 (with a higher limit it gives the same numerical result up to the 10th decimal place), we obtain that the maximum additional contribution will be proportional to $-8.6936754\text{x}10^{-10} - j9.96872766\text{x}10^{-11}$. This result is obtained by taking the cosine as a step and analytically solving the integral,

$$J(p,q) = \int_{3.9789}^{20} \left(\sqrt{\alpha^2 + j} - \alpha\right)e^{-5.0265\alpha}\, d\alpha \tag{28}$$

Thus, in practical terms it can be said that the error due to the change of limit is negligible.

E. Numerical Solution

For the analysis of the numerical implementation, we use the market formula in Table I; additionally, the results of Newton's first three closed formulas are included, colloquially called trapezoidal rule, Simpson's 1/3 rule and Simpson's 3/8 rule may be the most commonly used. Table II shows the maximum and minimum percentage difference of Newton Close Formulas, Table III shows the maximum and minimum percentage difference of Newton Open Formulas, Table IV shows the maximum and minimum percentage difference of Gauss Quadrature Formulas, and finally, Table V shows the maximum and minimum percentage difference of Gary Formulae, taking as reference the Gauss-Legendre Quadrature of order 15.

The results in Tables II to V were obtained as follows:

- A total of 101 formulations were applied: the 100 proposed formulations and Gary's formula.
- A parametric study was conducted on 10,000 cases, derived from variations in the values of p and q.
- Each case was solved using the 101 formulations, yielding a total of 1,010,000 integrals.
- The deviations for each case were determined using the Gauss-Legendre Quadrature of order 15 as a reference, computing the differences between the 100 solutions and this reference.

The procedure for constructing Tables II to V is as follows:

- The 10,000 cases were solved using the Gauss-Legendre Quadrature of order 15, and the results were stored in the matrix GQ15.

- The same cases were then solved using the closed Newton-Cotes formula of order 1 (trapezoidal rule), and the results were stored in the matrix CNC01.
- The percentage error between CNC01 and GQ15 was calculated for all cases, and the maximum and minimum error values were retained to define the error range on all the cases.
- This process was repeated for the remaining 99 formulations.

TABLE II
MAXIMUM PERCENT DIFFERENCE OF NEWTON CLOSE FORMULAS

NEWTON CLOSE FORMULAS				
Order	Self-real	Self imag	Mutual real	Mutual imag
1	0.0033478	0.00014031	0.0033478	0.00014047
2	5.8025e-08	2.1865e-09	5.7314e-08	2.2198e-09
3	1.306e-07	4.9215e-09	1.2904e-07	4.9931e-09
4	4.6293e-11	1.6924e-11	4.301e-11	3.9635e-11
6	4.5954e-11	1.7041e-11	4.819e-11	3.9479e-11
7	4.4835e-11	1.7197e-11	4.0123e-11	3.9349e-11
8	6.3022e-11	1.7093e-11	5.211e-11	3.9518e-11
10	1.2195e-10	1.8262e-11	8.9119e-11	3.8282e-11
33	0.97937	0.053251	0.75581	0.051771

MINIMUM PERCENT DIFFERENCE OF NEWTON CLOSE FORMULAS

NEWTON CLOSE FORMULAS				
Order	Self-real	Self imag	Mutual real	Mutual imag
1	1.0499e-06	9.4e-08	1.0499e-06	9.4107e-08
2	0	0	0	4.8763e-15
3	0	0	0	8.1272e-16
4	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
10	0	0	0	0
33	3.7718e-05	0.0018973	3.7654e-05	0.00040964

TABLE III
MAXIMUM PERCENT DIFFERENCE OF NEWTON OPEN FORMULAS

NEWTON OPEN FORMULAS				
Order	Self-real	Self imag	Mutual real	Mutual imag
4	1.8575e-10	1.6859e-11	1.8045e-10	3.9648e-11
6	3.0714e-10	1.7392e-11	2.5596e-10	3.9609e-11
7	1.6324e-10	1.7574e-11	1.4982e-10	3.9401e-11
8	1.278e-09	1.9704e-11	1.0913e-09	3.9596e-11
9	3.9346e-10	1.9769e-11	3.2978e-10	3.7125e-11
33	6.0736	0.32469	5.1289	0.32079

MINIMUM PERCENT DIFFERENCE OF NEWTON OPEN FORMULAS

NEWTON OPEN FORMULAS				
Order	Self-real	Self imag	Mutual real	Mutual imag
4	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
33	0.00075609	0.011954	0.0019754	0.002575

TABLE IV
MAXIMUM PERCENT DIFFERENCE OF GAUSS QUADRATURE FORMULAS

GAUSS LEGENDRE QUADRATURE				
Order	Self-real	Self imag	Mutual real	Mutual imag
6	5.3723e-11	2.5731e-11	8.9798e-11	9.819e-11
10	5.0624e-11	1.0677e-11	6.4536e-11	6.9855e-11
15	Reference	Reference	Reference	Reference
24	3.098e-11	8.7155e-12	5.1728e-11	3.6371e-11
27	3.3726e-11	9.1051e-12	5.0072e-11	3.7177e-11
33	4.1793e-11	1.082e-11	5.5931e-11	3.8087e-11

MINIMUM PERCENT DIFFERENCE OF GAUSS QUADRATURE FORMULAS

GAUSS LEGENDRE QUADRATURE				
Order	Self-real	Self imag	Mutual real	Mutual imag
6	0	0	0	0
10	0	0	0	0
15	Reference	Reference	Reference	Reference
24	0	0	0	0
27	0	0	0	0
33	0	0	0	0

TABLE V
MAXIMUM PERCENT DIFFERENCE OF GARY FORMULAE

GARY FORMULAE				
	Self-real	Self imag	Mutual real	Mutual imag
Gary	2.3987	1.1258	3.8347	1.1271

MINIMUM PERCENT DIFFERENCE OF GARY FORMULAE

GARY FORMULAE				
	Self-real	Self imag	Mutual real	Mutual imag
Gary	0.029435	0.00039762	7.3566e-05	1.0174e-05

Figure 2 shows the behavior of the real and imaginary part of the ground impedance for both self-impedance and mutual impedance. It can be seen how the behavior is similar in all cases and is very smooth, with a well-defined curvature. Figure 3 shows the differences between Gary's formulas and Gauss Quadrature of order 15; It can be clearly noted how the difference of the imaginary parts follows the same behavior of the ground impedance; On the other hand, the real part has a kind of peak where the differences are maximum; here, the combined parameters can be given for non-feasible physical situations or, on the contrary, for feasible physical situations. So, this situation must be taken into account when using any formulation.

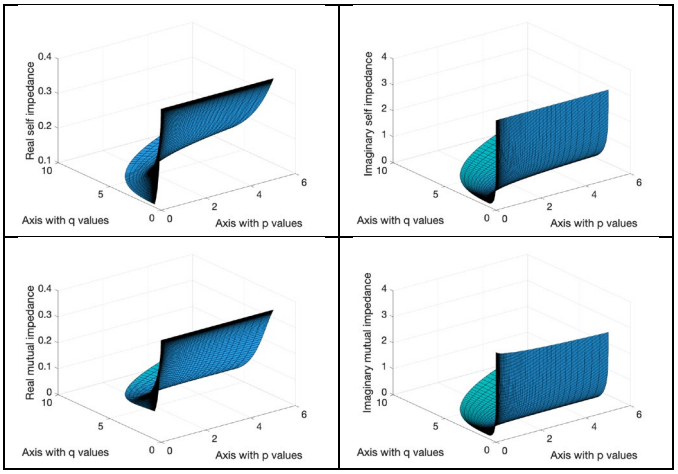

Fig. 2. Graph of self and mutual impedance calculated with Gary's formulas.

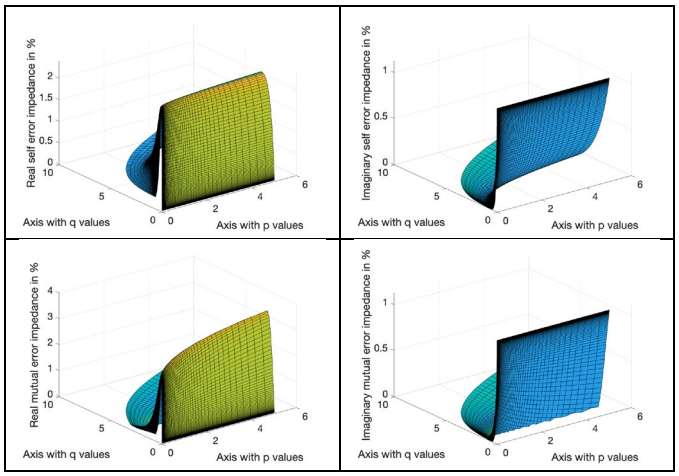

Fig. 3. Differences in self and mutual impedance calculated with Gary's formulas and Gauss quadrature order 15.

IV. Pollaczek Integrals

The self and mutual impedance for a cable is denoted in terms of the Pollaczek integral as:

$$Z_{E,ii} = \frac{j\omega\mu}{2\pi}\left[K_0\left(\gamma_g d\right) - K_0\left(\gamma_g D\right) + J_s\right] \tag{29a}$$

$$Z_{E,ik} = \frac{j\omega\mu}{2\pi}\left[K_0\left(\gamma_g d\right) - K_0\left(\gamma_g D\right) + J_m\right], \tag{29b}$$

where

$$J_s = \int_{-\infty}^{+\infty}\left[\left(e^{-2h\sqrt{\alpha^2+\gamma^2}}\right)\Big/\left(\alpha+\sqrt{\alpha^2+\gamma^2}\right)\right]e^{j\alpha r}\,d\alpha$$

$$J_m = \int_{-\infty}^{+\infty}\left[\left(e^{-\left(h_i+h_k\right)\sqrt{\alpha^2+\gamma^2}}\right)\Big/\left(\alpha+\sqrt{\alpha^2+\gamma^2}\right)\right]e^{j\alpha x}d\alpha$$

Accordingly, with [22] the lower limit of the integral could be replaced by 0 making a rigid analysis to obtain the same result; that means, this change does not affect the Pollaczek integral. So, the integrals could be denoted as follows:

$$J_s = \int_{0}^{+\infty}\left[\left(e^{-2h\sqrt{\alpha^2+\gamma^2}}\right)\Big/\left(\alpha+\sqrt{\alpha^2+\gamma^2}\right)\right]\left(e^{j\alpha r}+e^{-j\alpha r}\right)\cdot d\alpha$$

$$J_m = \int_{0}^{+\infty}\left[\left(e^{-\left(h_i+h_k\right)\sqrt{\alpha^2+\gamma^2}}\right)\Big/\left(\alpha+\sqrt{\alpha^2+\gamma^2}\right)\right]\left(e^{j\alpha x}+e^{-j\alpha x}\right)d\alpha$$

In the same way, in [26] it is proposed to change the exponentials by the cosine function, so one obtains:

$$J_s = 2\int_{0}^{+\infty}\left[\left(e^{-2h\sqrt{\alpha^2+\gamma^2}}\right)\Big/\left(\alpha+\sqrt{\alpha^2+\gamma^2}\right)\right]\cos\left(r\alpha\right)d\alpha$$

$$J_m = 2\int_{0}^{+\infty}\left[\left(e^{-\left(h_i+h_k\right)\sqrt{\alpha^2+\gamma^2}}\right)\Big/\left(\alpha+\sqrt{\alpha^2+\gamma^2}\right)\right]\cos\left(x\alpha\right)d\alpha$$

The solution proposed by Saad, Gaba & Giroux [22] is,

$$Z_E^{ii} = \frac{j\omega\mu}{2\pi}\left[K_0\left(\gamma_g R\right) + \frac{2}{4+\gamma_g^2 R^2}e^{-2h\gamma}\right] \tag{30a}$$

$$Z_E^{ik} = \frac{j\omega\mu}{2\pi}\left[K_0\left(\gamma_g d\right) + \frac{2}{4+\gamma_g^2 x^2}e^{-\left(h_i+h_k\right)\gamma}\right] \tag{30b}$$

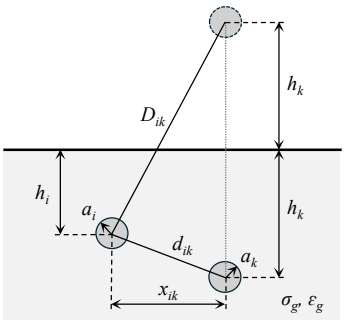

Fig. 4. Cable geometry (adapted from [27]).

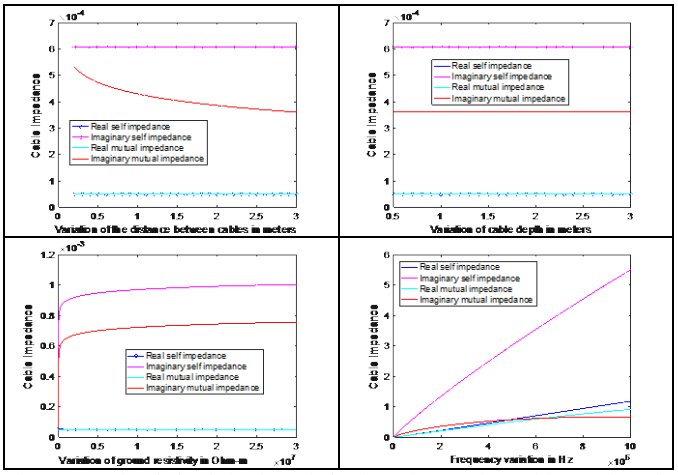

Fig. 5. Graph of the behavior of the impedance of a cable varying the parameters one by one.

The geometry of a pair of cables is shown in figure 4. This figure shows all the parameters on which the impedance of the cable depends. If the parameters are varied individually, the results are shown in Figure 5. This figure shows how the impedance calculated with (22) changes by varying each of the parameters.

A. Superior Limit of the Pollaczek Integrals

By varying each parameter, a sensitivity study is carried out to detect the worst scenario or the most critical condition. In this way, it is found that the Pollaczek function for self and mutual impedance has its worst behavior as shown in Figure 6.

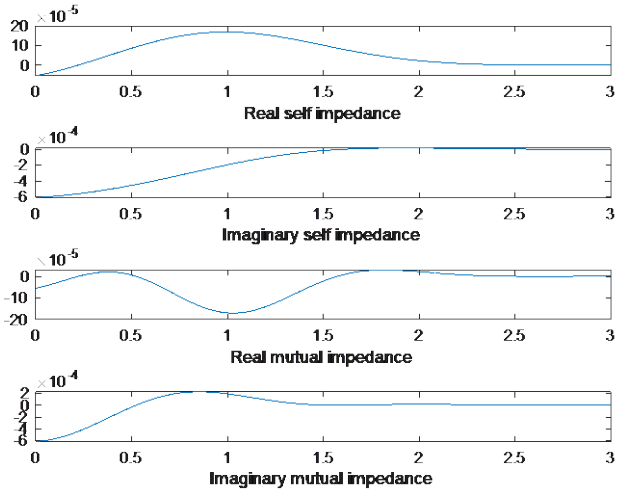

Fig. 6. Graph of the behavior of the Pollaczek integrals.

With this result we obtain that the upper limit of the integral can be 3 without affecting the results. Numerical tests are still done with higher limits but there is no numerical effect and for that reason in the end 3 is adopted as a good upper limit. It should be noted that although the upper limit does not affect the results, the number of samples does have an effect; for this reason, many tests were done, obtaining that with 6 thousand samples per unit was where the best results were obtained, in case we have more than 3 cycles, we will use the rule of 6 thousand samples per cycle.

B. Choose the parameter to be varied

Analyzing the behavior of the Pollaczek integral shown in figure 5; the two parameters, that have the most effect on the function, are chosen and these are the resistivity of the ground and the frequency. We varying these parameters logarithmically, taking 100 values distributed between the minimum and the maximum. For resistivity, the minimum is taken as 10^1 and the maximum as 10^2. In the case of frequency, 10^1 is taken as the minimum and 10^4 as the maximum. With these values the calculations are made, obtaining 10000 results.

C. Numerical Solution

The impedance of a cable is done numerically using the same formulas as Newton's closed, Newton's open, and Gauss-Legendre Quadrature. As for the approximate formula, the one proposed by Saad, Gaba and Giroux is used. Table VI shows the results obtained with Newton's closed formulas, here it is shown how the formula of order 33 gives results with maximum significant differences. Table VII shows that Newton's open formulas of order 33 have maximum significant differences. Table VIII shows how the Gauss-Legendre Quadrature tends to improve when the order increases, considering that the order 15 of the Gauss-Legendre Quadrature is taken as a reference. In the case of the Saad, Gaba & Giroux formulas (Table IX), it is shown that the maximum percentage difference is significant.

TABLE VI
MAXIMUM PERCENT DIFFERENCE OF NEWTON CLOSE FORMULAS

NEWTON CLOSE FORMULAS				
Order	Self-real	Self imag	Mutual real	Mutual imag
1	0.00672742	0.00022632	0.007368506	0.000437290
2	0.00280493	0.000223858	0.003072221	0.000432518
3	0.00359482	0.00024867	0.003937390	0.000480465
4	0.00241593	0.000226013	0.002646160	0.000436680
6	0.00211917	0.000217444	0.002321117	0.000420124
7	0.00229973	0.000226675	0.002518885	0.000437960
8	0.00190761	0.000208939	0.002089400	0.000403693
10	0.00175005	0.000200004	0.001916827	0.000386429
33	7.51905924	0.840371270	8.235580706	1.623675522

MINIMUM PERCENT DIFFERENCE OF NEWTON CLOSE FORMULAS

NEWTON CLOSE FORMULAS				
Order	Self-real	Self imag	Mutual real	Mutual imag
1	0	0	2.68087e-21	1.94791e-22
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
10	0	0	0	0
33	9.4282e-14	2.4357e-14	2.7712e-15	1.15133e-15

TABLE VII
MAXIMUM PERCENT DIFFERENCE OF NEWTON OPEN FORMULAS

NEWTON OPEN FORMULAS				
Order	Self-real	Self imag	Mutua real	Mutua imag
4	0.00854035	0.000963404	0.009354190	0.001861396
6	0.00763402	0.000986660	0.008361495	0.001906330
7	0.00795502	0.000998508	0.008713081	0.001929222
8	0.00872198	0.000985963	0.009553129	0.001904984
9	0.00853355	0.000997117	0.009346749	0.001926534
33	5.70105996	0.684765404	6.244330266	1.323185144

MINIMUM PERCENT DIFFERENCE OF NEWTON OPEN FORMULAS

NEWTON OPEN FORMULAS				
Order	Self-real	Self imag	Mutua real	Mutua imag
4	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
33	6.0105e-13	1.5561e-13	1.67456e-14	7.84317e-15

TABLE VIII
MAXIMUM PERCENT DIFFERENCE OF GAUSS QUADRATURE FORMULAS

GAUSS LEGENDRE QUADRATURE				
Order	Self-real	Self imag	Mutual real	Mutual imag
6	0.00026062	3.0392e-05	0.000285460	5.87204e-05
10	9.6076e-05	1.13796e-05	0.000105231	2.19866e-05
15	Reference	Reference	Reference	Reference
24	5.9103e-05	7.02756e-06	6.47353e-05	1.35779e-05
27	6.4537e-05	7.79802e-06	7.06869e-05	1.50665e-05
33	7.0912e-05	8.43301e-06	7.76700e-05	1.62934e-05

MINIMUM PERCENT DIFFERENCE OF GAUSS QUADRATURE FORMULAS

GAUSS LEGENDRE QUADRATURE				
Order	Self-real	Self imag	Mutual real	Mutual imag
6	0	0	0	0
10	0	0	0	0
15	Reference	Reference	Reference	Reference
24	0	0	0	0
27	0	0	0	0
33	0	0	0	0

TABLE IX
MAXIMUM PERCENT DIFFERENCE OF SGG FORMULAE

SAAD, GABA & GIROUX FORMULAE				
	Self-real	Self imag	Mutual real	Mutual imag
SGG	2.84799547	0.523055309	4.393381345	1.025565876

MINIMUM PERCENT DIFFERENCE OF SGG FORMULAE

SAAD, GABA & GIROUX FORMULAE				
	Self-real	Self imag	Mutual real	Mutual imag
SGG	5.2149e-11	5.0974e-11	1.0394e-11	2.44846e-12

Figure 7 shows the behavior of the real and imaginary part of both the self-impedance and the mutual-impedance. It can be seen with the naked eye that there is a sector that behaves atypically; this occurs when the earth's resistivity values are low, and the behavior occurs throughout the entire frequency range. Likewise, figure 8 shows the differences between the Saad, Gaba & Giroux formulas and the Gauss Quadrature; there it is clearly noted that the differences occur precisely in the

same sector where the behavior of the results is atypical or with abrupt variation. The rest of the results, that is, the majority have an almost linear behavior and there the differences are almost imperceptible and tend towards zero.

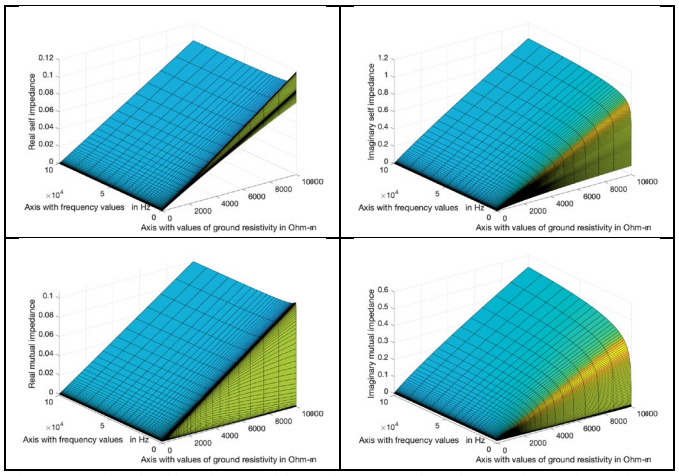

Fig. 7. Graph of self and mutual impedance Saad, Gaba & Giroux formulas.

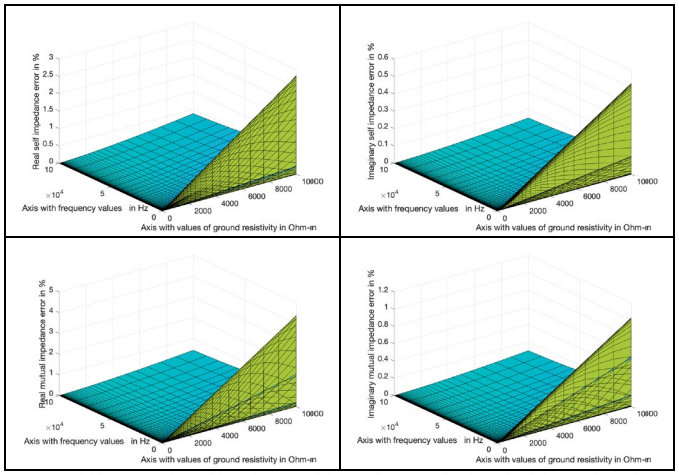

Fig. 8. Differences in self and mutual impedance calculated with Saad, Gaba & Giroux formulas and Gauss quadrature order 15.

V. Conclusions

In numerical analysis, no method can be universally extrapolated. However, experimental evidence suggests that numerical methods should yield comparable results when applied to similar functions. Therefore, numerical rules of a given order can be applied to functions lacking simple analytical solutions. Nonetheless, there is no exact reference to verify whether behaviors observed in one set of functions apply similarly to Carson and Pollaczek integrals. Consequently, it cannot be concluded that the rule taken as a reference is the most precise, but neither can this possibility be dismissed. Even though the conclusions are not absolute, it is worth noting that if a group of formulas were used, the results would likely remain consistent, as the maximum percentage differences among many numerical methods are negligible. Despite the principle that one numerical method should not serve as a substitute for another, if multiple integration rules—derived from distinct methodologies and implementations—produce similar or equal results, then these numerical methods can be considered reliable.

Based on our findings, we conclude the following:

- A homogeneous distribution of information is not necessarily optimal. For the analyzed integrals, Newton-Cotes methods perform worse than Gauss-Legendre methods.
- More information does not always lead to better accuracy. There is a limit on the number of points beyond which a given rule ceases to improve.
- High-order Newton-Cotes rules are not the best choice, but neither are low-order rules.
- Newton-Cotes coefficients for high-order rules are nearly impossible to obtain exactly with 64-bit computing precision.
- The complexity of the Carson integral arises from defining its upper limit, which varies across cases. A specialized program was developed to adapt to each case.
- Each Carson integral was evaluated with 6000 samples per cycle, with a variable number of samples depending on the case.
- The integral's upper limit was constrained, and the maximum numerical error was assessed, confirming that the calculated upper limit was appropriate.
- The maximum differences between Gary's formulas and numerical methods occur under unpredictable parameter combinations. While these differences can appear in practical scenarios, they are not significant enough to justify replacing Gary's formulas with a numerical method for all cases.
- For the Pollaczek integral, around 18,000 samples per cycle yielded the highest correlation. Testing with both more and fewer samples degraded the results. This conclusion was reached using the Saad, Gaba & Giroux formulas as a reference.
- Due to numerical conditioning, the real part of both self-impedance and mutual impedance exhibits lower numerical precision in all cases.
- The formulas proposed by Saad, Gaba & Giroux generally align with numerical methods, with negligible differences. Therefore, they are a suitable approach for calculating cable impedance. However, in cases where maximum differences occur, using a numerical method is advisable.

VI. References

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